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Hermite radial basis function (HRBF). Gradient and matrix (summary)

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1 NOTATIONS

- $\mathbf{v}$: vectors are in bold we write their componants like this $\mathbf{v} = [v_{x_1}, \dots, v_{x_d}]^T$ in dimension d.
- $\mathbf{v}^T$: is a row vector (and $\mathbf{v}$ a column vector).
- ∇v : is a column vector.
- $\mathbf{M}$: Matrices are in bold capital.
- $\mathbf{v}_0 \otimes \mathbf{v}_1$: tensor product between $\mathbf{v}_0$ and $\mathbf{v}_1$ (outer product).

2 HRBF

Here the vector $\mathbf{x}$ dimension is d. In the specific case of a implicit surface $\mathbf{x} \in \mathbb{R}^3$ is a point of the ambiant space and $d = 3$.

$$\begin{aligned} f(\mathbf{x}) &= \sum_i^N \alpha_i \phi(\|\mathbf{x} - \mathbf{p}_i\|) + \nabla \left[\phi(\|\mathbf{x} - \mathbf{p}_i\|)\right]^T \boldsymbol{\beta}_i \\ &= \sum_i^N \alpha_i \phi(\|\mathbf{x} - \mathbf{p}_i\|) + \mathbf{a}_i(\mathbf{x})^T \boldsymbol{\beta}_i \end{aligned}$$

∇f is the column vector of dimension d.

$$\nabla f(\mathbf{x}) = \sum_i^N \alpha_i \mathbf{a}_i(\mathbf{x}) + \underbrace{(\mathbf{B}_i(\mathbf{x}) + \mathbf{C}_i(\mathbf{x}))}_{\text{Hessian of } \phi(\|\mathbf{x} - \mathbf{p}_i\|)} \boldsymbol{\beta}_i$$

$$\mathbf{a}_i(\mathbf{x}) = \phi'(\|\mathbf{x} - \mathbf{p}_i\|) \frac{\mathbf{x} - \mathbf{p}_i}{\|\mathbf{x} - \mathbf{p}_i\|}$$

$$\mathbf{B}_i(\mathbf{x}) = \begin{pmatrix} b_i(\mathbf{x}) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & b_i(\mathbf{x}) \end{pmatrix} \text{ Diagonal matrix of dimension } d \times d \text{ with } b_i(\mathbf{x}) = \frac{\phi'(\|\mathbf{x} - \mathbf{p}_i\|)}{\|\mathbf{x} - \mathbf{p}_i\|}$$

$$\mathbf{C}_i(\mathbf{x}) = c_i(\mathbf{x}).\big((\mathbf{x} - \mathbf{p}_i) \otimes (\mathbf{x} - \mathbf{p}_i)\big) = \begin{pmatrix} (x_1 - p_{i,x_1})(x_1 - p_{i,x_1}) c_i(\mathbf{x}) & \dots & (x_d - p_{i,x_d})(x_1 - p_{i,x_1}) c_i(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ (x_1 - p_{i,x_1})(x_d - p_{i,x_d}) c_i(\mathbf{x}) & \dots & (x_d - p_{i,x_d})(x_d - p_{i,x_d}) c_i(\mathbf{x}) \end{pmatrix}$$

$$c_i(\mathbf{x}) = \frac{1}{\left(\|\mathbf{x}-\mathbf{p}_i\|\right)^2}\left(\phi''(\|\mathbf{x}-\mathbf{p}_i\|) - \frac{\phi'(\|\mathbf{x}-\mathbf{p}_i\|)}{\|\mathbf{x}-\mathbf{p}_i\|}\right)$$

3 Representation matricielle

Find the weights α_i and $\boldsymbol{\beta}_i$ of the HRBF is done by solving the linear system below with a LU decomposition:

$$\begin{pmatrix} f(\mathbf{p}_i) \\ \nabla f(\mathbf{p}_i) \end{pmatrix} = \begin{pmatrix} c \\ \mathbf{n}_i^T \end{pmatrix}$$

The equation system $\mathbf{A}.\mathbf{x}^T = \mathbf{b}^T$ can be written:

$$\underbrace{\begin{pmatrix} \mathbf{f}_1^T(\mathbf{x}_1) & \dots & \mathbf{f}_N^T(\mathbf{x}_1) \\ \nabla f_1(\mathbf{x}_1) & \dots & \nabla f_N(\mathbf{x}_1) \\ \vdots & \ddots & \vdots \\ \mathbf{f}_1^T(\mathbf{x}_N) & \dots & \mathbf{f}_N^T(\mathbf{x}_N) \\ \nabla f_1(\mathbf{x}_N) & \dots & \nabla f_N(\mathbf{x}_N) \end{pmatrix}}_{\text{Matrix of dimension } (N.(d+1))\times(N.(d+1))} \begin{pmatrix} \alpha_1 \\ \boldsymbol{\beta}_1 \\ \vdots \\ \alpha_N \\ \boldsymbol{\beta}_N \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{n}_1 \\ \vdots \\ 0 \\ \mathbf{n}_N \end{pmatrix}$$

Note : $\boldsymbol{\beta}_i$ is the column vector of dimension d.

$$\mathbf{f}_i^T(\mathbf{x}) = \underbrace{\begin{pmatrix} \phi(\|\mathbf{x}-\mathbf{p}_i\|) & \underbrace{\mathbf{a}_i^T(\mathbf{x})}_{\text{dimension } d} \end{pmatrix}}_{\text{row vector of dimension } d+1}$$

$$\nabla f_i(\mathbf{x}) = \underbrace{\begin{pmatrix} \mathbf{a}_i(\mathbf{x}) & \mathbf{B}_i(\mathbf{x}) + \mathbf{C}_i(\mathbf{x}) \end{pmatrix}}_{\text{Matrix of dimension } d\times(d+1)}$$

In practice when filling the matrix, every coefficient with division by zero (sometime the norm is null) must be replaced by a null value. We detail the coefficients of a single matrix block $(d+1)\times(d+1)$ of $\mathbf{A}$:

$$\begin{pmatrix} \mathbf{f}_i^T(\mathbf{x}_j) \\ \nabla f_i(\mathbf{x}_j) \end{pmatrix} = \begin{pmatrix} \phi(\|\mathbf{x}_j-\mathbf{p}_i\|) & \frac{\phi'(\|\mathbf{x}_j-\mathbf{p}_i\|)}{\|\mathbf{x}_j-\mathbf{p}_i\|}(x_{j,x_1}-p_{i,x_1}) & \dots & \frac{\phi'(\|\mathbf{x}_j-\mathbf{p}_i\|)}{\|\mathbf{x}_j-\mathbf{p}_i\|}(x_{j,x_d}-p_{i,x_d}) \\ \phi'(\|\mathbf{x}_j-\mathbf{p}_i\|)\frac{x_{j,x_1}-p_{i,x_1}}{\|\mathbf{x}_j-\mathbf{p}_i\|} & (x_{j,x_1}-p_{i,x_1})(x_{j,x_1}-p_{i,x_1})c_i(\mathbf{x}_j)+b_i(\mathbf{x}_j) & \dots & (x_{j,x_d}-p_{i,x_d})(x_{j,x_1}-p_{i,x_1})c_i(\mathbf{x}_j) \\ \vdots & \vdots & \ddots & \vdots \\ \phi'(\|\mathbf{x}_j-\mathbf{p}_i\|)\frac{x_{j,x_d}-p_{i,x_d}}{\|\mathbf{x}_j-\mathbf{p}_i\|} & (x_{j,x_1}-p_{i,x_1})(x_{j,x_d}-p_{i,x_d})c_i(\mathbf{x}_j) & \dots & (x_{j,x_d}-p_{i,x_d})(x_{j,x_d}-p_{i,x_d})c_i(\mathbf{x}_j)+b_i(\mathbf{x}_j) \end{pmatrix}$$