M1 – International

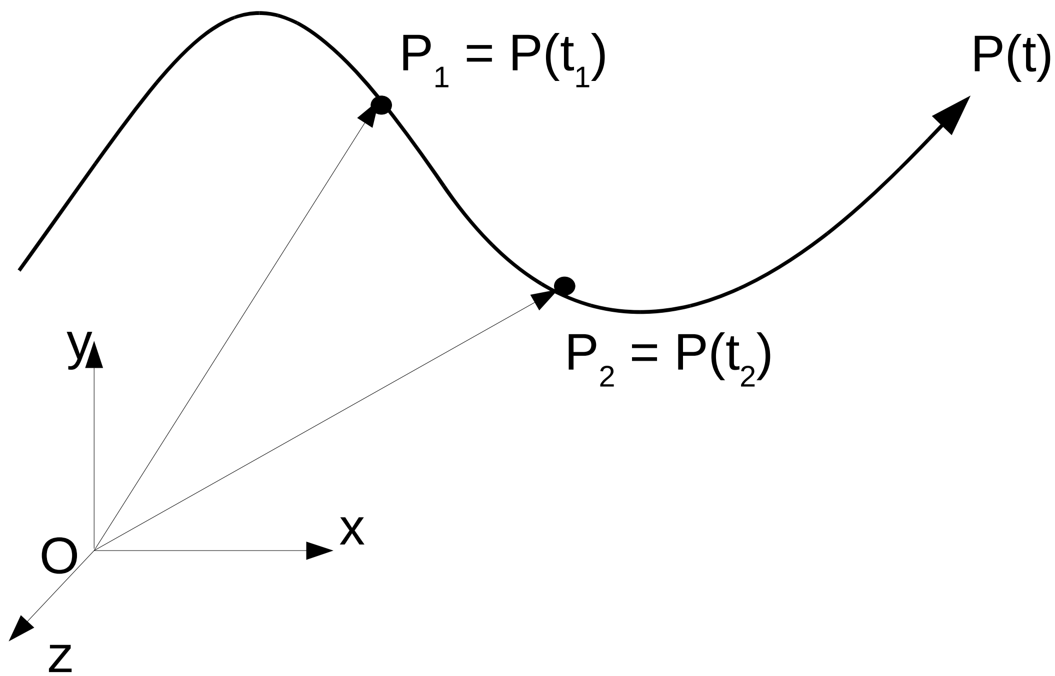
Differential Geometry
and
Bézier curves

Plan

- Introduction
- Parametric equation of curves
- Parametric equations of a line
- Differential geometry
  - Tengent, principal normal, Frénet frame
  - Curvature, torsion
- Bézier curves

Basics

- A parametric curve can be seen as the trajectory of a point P moving in space. The parameter is then the time **t** even though any parameter **u** can be used in practice.



- Note that the point P(x,y,z) has the same coordinates as vector $\overrightarrow{OP}$

Definition

- A parametric curve in the R^3 space is defined by a function

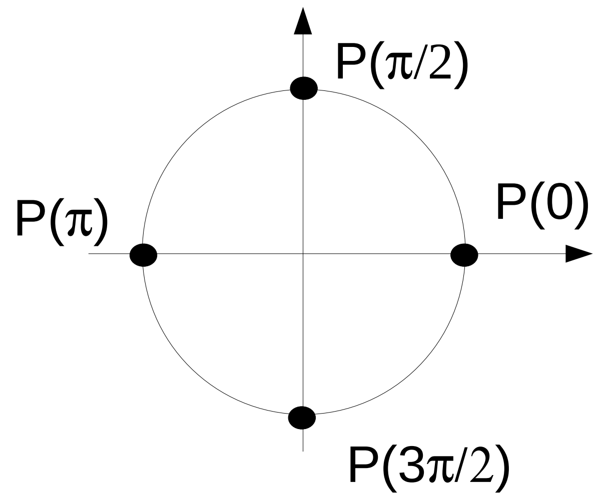
$$
\begin{aligned}
f: \quad & \mathbb{R} \quad \rightarrow \quad \mathbb{R}^3 \\
& u \quad \rightarrow \quad P(u) = \begin{cases} x(u) = f_x(u) \\ y(u) = f_y(u) \\ z(u) = f_z(u) \end{cases}
\end{aligned}
$$

- Thus, for each value of parameter **u**, we compute independently each of the three corrdinates x,y, z of the point P(u)
- The same curve can have several different parametric equations (possibly an infinity).

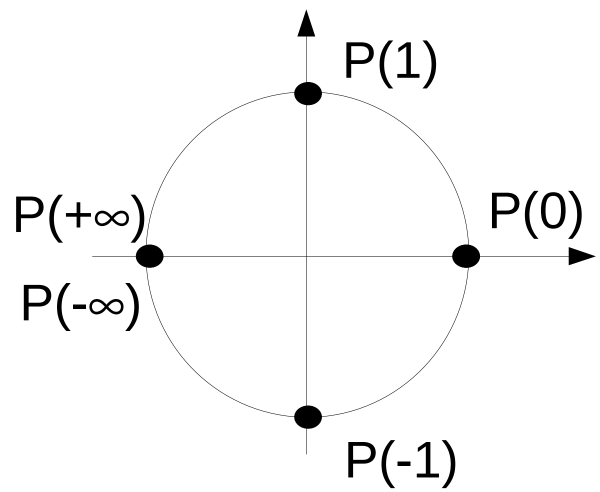
Example

- Parametric equations of a circle in R² :

$$P(u):\begin{cases} x(u)=r\cos u \\ y(u)=r\sin u \end{cases} \qquad u\in[0,2\pi[$$



$$P(u):\begin{cases} x(u)=r\dfrac{1-u^2}{1+u^2} \\ y(u)=r\dfrac{2u}{1+u^2} \end{cases} \qquad u\in]-\infty,+\infty[$$

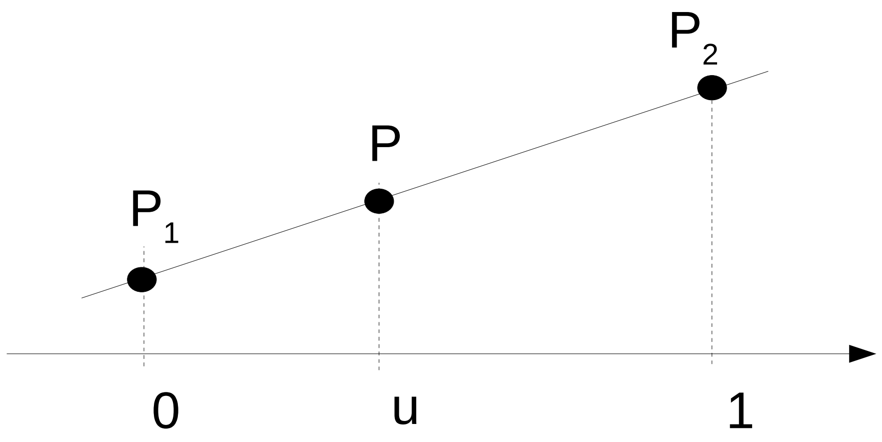


Parametric equation of a line

- Parametric equation of a line in R^3 passing by two points P_1 et P_2 :

$$P(u)=(1-u)P_1+uP_2 \quad \equiv \quad P(u)\begin{cases} x(u)=(1-u)x_1+u\,x_2 \\ y(u)=(1-u)y_1+u\,y_2 \\ z(u)=(1-u)z_1+u\,z_2 \end{cases} \quad u\in R$$

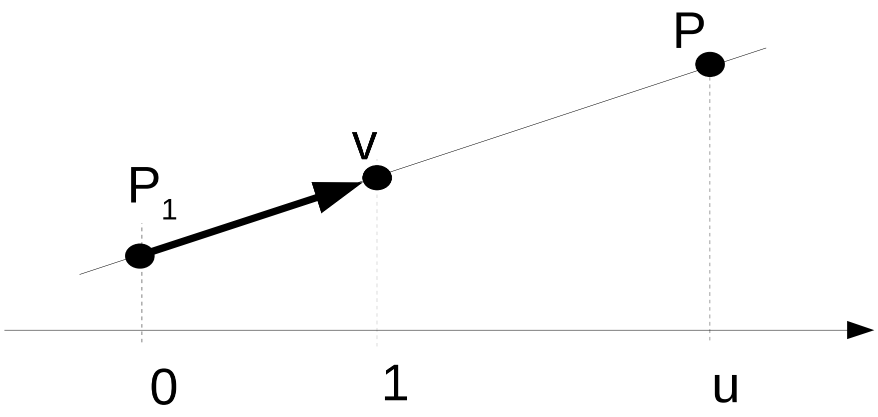
- This equation leads to the notion of linear interpolation. Indeed, when u varies between 0 et 1, the point P linearly covers the segment from P_1 to P_2



Parametric equation of a line

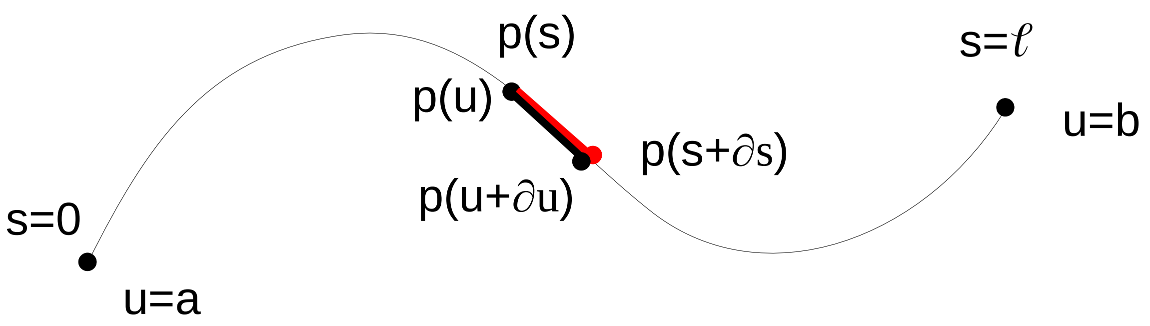
- A line can also be defined with a point P_1 and a vector v :

$$P(u) = P_1 + u\,v$$



Differential geometry : parameters

- Random parameter : u ∈ [a,b]   curvilinear abscissa : s ∈ [0,ℓ].
  - s is the lenght of curve from the origine to the point P(s):



Tangent vector

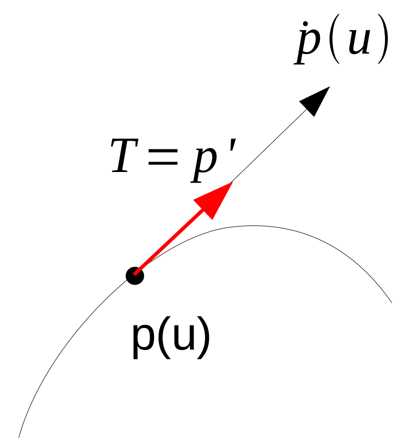
- Unit tangent vector :

$$T = \frac{dp}{ds} = p'$$

$$T = \lim_{\partial s \to 0} \frac{p(s+\partial s) - p(s)}{\partial s}$$

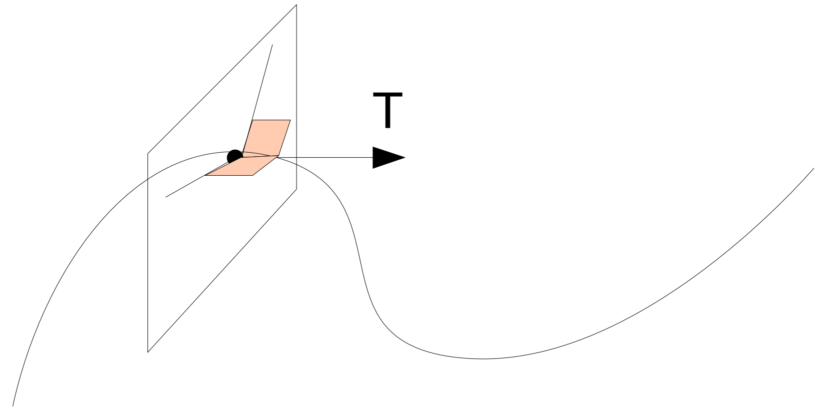
  - Practical computation :

$$\boxed{\dot{p}(u) = \frac{dp}{du} = T\,\|\dot{p}(u)\|}$$



Principal normal vector

- The tangent is known. How to define a orthogonal vector with a predictable orientation ?



- Exercise :
  - Show that the derivative of a vector with constant norm is orthogonal to that vector :

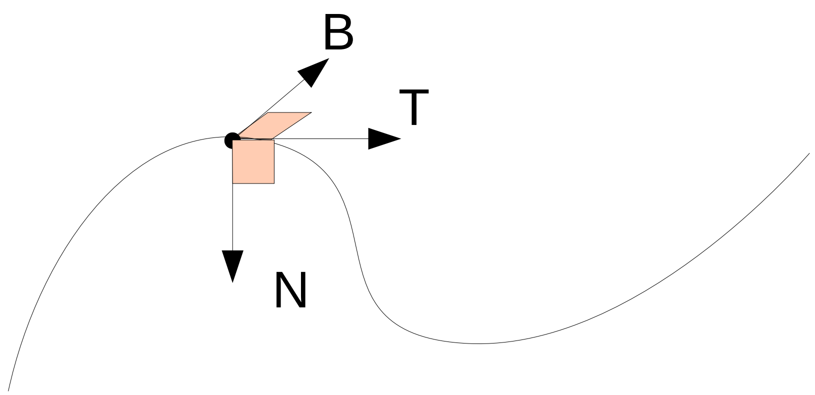
    we have $\forall u : v(u).v(u) = \|v\|^2$ show that $\dot{v}(u) \perp v(u)$

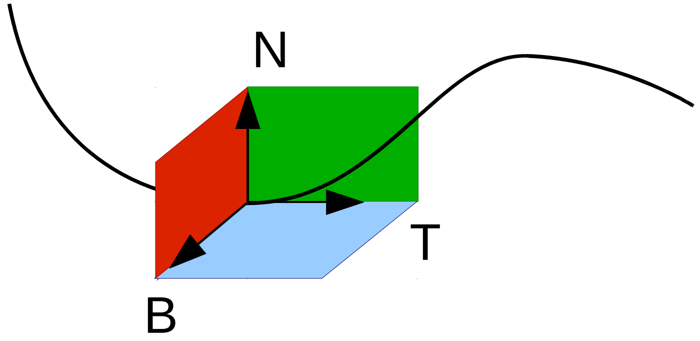
- The principal normal vector **N** is defined from the derivative of the unit tangent vector :

$$T = \frac{dP}{ds} \qquad \|T\| = \left\|\frac{dP}{ds}\right\| = 1 \,\forall u \qquad v = \frac{dT}{ds} = \frac{d^2P}{ds^2} \qquad \boxed{N = \frac{v}{\|v\|}}$$

Frénet frame

- It is a local frame in a point on a curve :
- Once the vectors T, N and B evaluated at point a P, the Frénet frame is the frame (T,N,B),centered in P.





- **(T,N)** defines the osculating plane
- **(B,T)** defines the tangent plane
- **(N,B)** defines the normal plane

Computation of the Frénet frame

- Unit tangent vector **T** and speed vector **v** :

$$v=\frac{dp}{du}=\dot{p}=\left[\frac{dx}{du},\frac{dy}{du},\frac{dz}{du}\right]$$

$$\boxed{T=\frac{\dot{p}}{\|\dot{p}\|}}$$

- Unit binormal vector **B** and acceleration vector **a** :

$$a=\frac{d^2p}{du^2}=\ddot{p}$$

$$\boxed{B=\frac{\dot{p}\wedge\ddot{p}}{\|\dot{p}\wedge\ddot{p}\|}}$$

- Principal normal vector :

$$\boxed{N=B\wedge T}$$

- Vector **N** points in the direction of the center of curvature, thus, when passing an inflexion point, the frame « flips ».

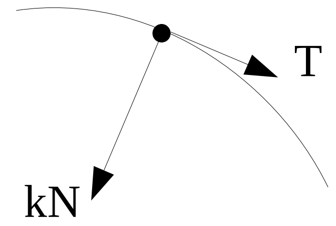
Curvature

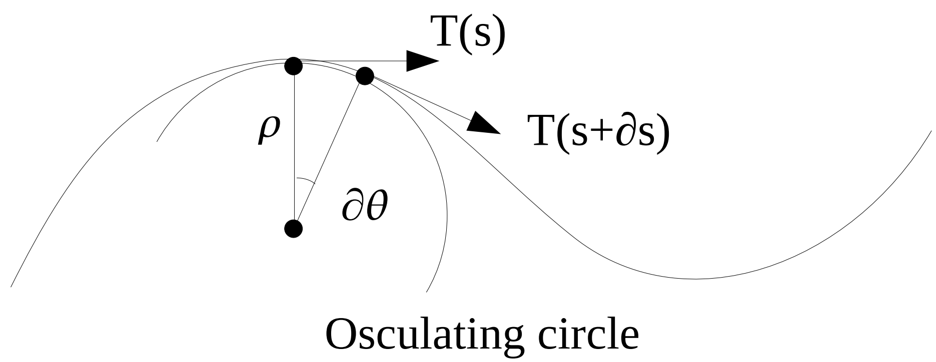
.

- By definition, the principal normal vector is :

$$\frac{d^2 p}{ds^2} = \frac{dT}{ds} = kN$$ where k is the curvature in p

$$k = \frac{1}{\rho}$$ where ρ is the curvature radius





Curvature

- Curvature :

$$(1) \qquad k = \frac{\| \dot{p} \wedge \ddot{p} \|}{\| \dot{p} \|^3}$$

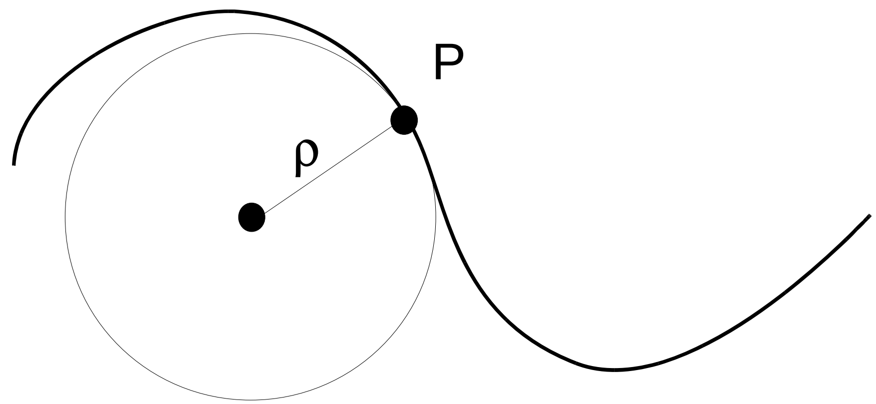
- Curvature vector :

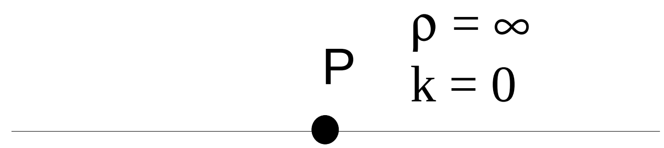
$$(2) \qquad k\,N = \frac{\dot{p} \wedge \ddot{p} \wedge \dot{p}}{\| \dot{p} \|^4}$$

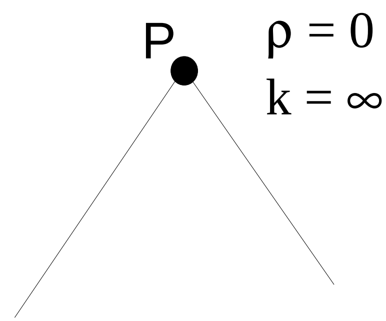
- Curvature radius :

$$\rho = \frac{1}{k}$$

- Inflexion point => k=0
- k=0 $\not\Rightarrow$ inflexion point







- Exercise
  - Show that expressions (1) and (2) correspond.

Torsion

- Torsion vector : $$\frac{dB}{ds} = \tau\, N$$

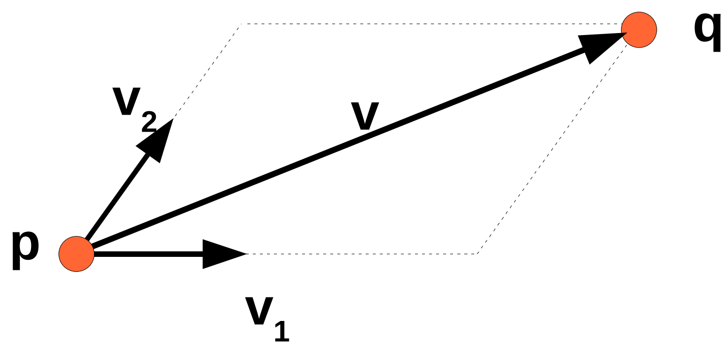
- Torsion : it is defined from the variation of the binormal vector.

$$\tau = \frac{\dot{p} \,.\, \left(\ddot{p} \wedge \dddot{p} \right)}{\left\| \dot{p} \wedge \ddot{p} \right\|^2}$$

- If the torsion is zero over the curve, the curve is planar.

Affine combination of points and parametric curves

Affine space



- A is an affine space. We can chose a point p in A and a set of n vectors $v_1,\ldots,v_n$ representing a bases of the vector space V. This any point q of A has a unique representation:

$$q = p + v_1 x_1 + \ldots + v_n x_n$$

- Column of coordinates $x = [x_1 \ldots x_n]^t \in \mathbb{R}^n$ represents the point q with respect to the affine frame p, v_1, ..., v_n.

Affine combination

- Any point q can be written as :

$$q = p_0 + (p_1 - p_0)x_1 + \ldots + (p_n - p_0)x_n$$

- And thus as :

$$q = p_0 x_0 + \ldots + p_n x_n$$

- Exercise :
  - Show that : $x_0 + \ldots + x_n = 1$

- Coefficients $\mathbf{x_i}$ are called **barycentric coordinates** of **q** with respect to the frame $\mathbf{p_0,\ldots,p_n}$.

Weighted sum

- Weighted sum of points :

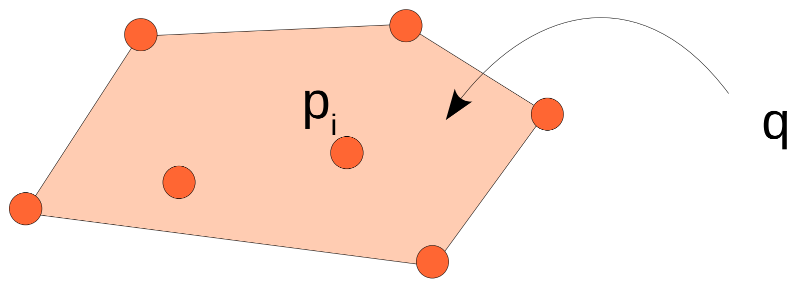
$$q = \sum p_i \alpha_i$$

  - **q** is a **point** if $\sum \alpha_i = 1$ and **q** is a **vector** if $\sum \alpha_i = 0$. Otherwise, **q is not defined**

- If the sum of weights is one : $\left(\sum \alpha_i = 1\right)$ then $q = \sum p_i \alpha_i$ is called an **affine combination**

- Moreover, if all weights α_i are positive then it is a **convex combination**

  - The point **q** is in the convex hull of points $\mathbf{p_i}$.



Affine combinaison and parametric curves

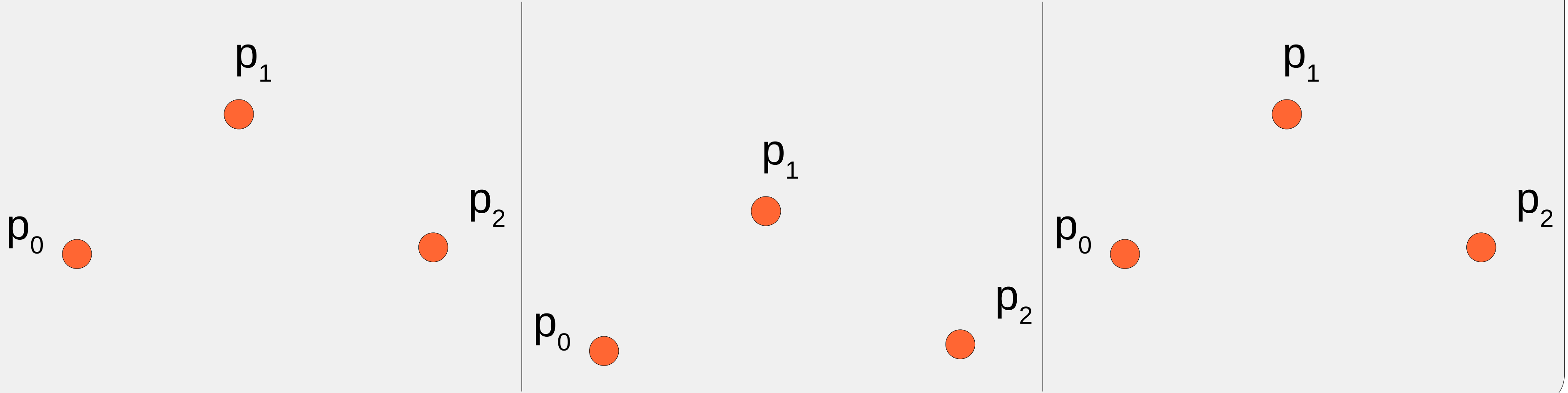
- Exercise :
  - 3 points p_0,p_1,p_2 and a function p(u) of equation :

$$p(u)=u^2 p_0+u\, p_1+p_2 \qquad u\in[0,1]$$

  - Plot p(u) when the frame origin is p_0, then when the origin is p_2 . What can you observe ? Why is it like this ? Does this function define a curve ?
  - Same questions with p(u) defined with the following equation :

$$p(u)=(1-u)^2 p_0+2u(1-u)\, p_1+u^2 p_2 \qquad u\in[0,1]$$

  - Show that the points of this second equation are in the convex hull of points p_i.



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Affine combinaison and parametric curves

$$\boxed{p(u)=\sum_{i=0}^{n} N_i^d(u)\,P_i} \qquad u\in[a,b]$$

with

$$\boxed{\sum_{i=0}^{n} N_i^d(u)=1} \qquad \forall u\in[a,b]$$

- A point of the curve is an affine combination of the control points P_i

  Thus, with respect to the control points, the relative curve shape and position remain invariant by affine transformations.

Bézier curves

Bernstein polynomial

- Binomial expension :

$$1=\left(u+\left(1-u\right)\right)^{n}=\sum_{i=0}^{n}\binom{n}{i}u^{i}\left(1-u\right)^{n-i}$$

This gives us the sum of n+1 polynomials of degree n called : **Bernstein polynomials :**

$$\boxed{B_i^n(u)=\binom{n}{i}u^{i}\left(1-u\right)^{n-i}\quad,\qquad i=0,\dots,n}$$

where

$$\binom{n}{i}=\frac{n!}{i!(n-i)!}$$

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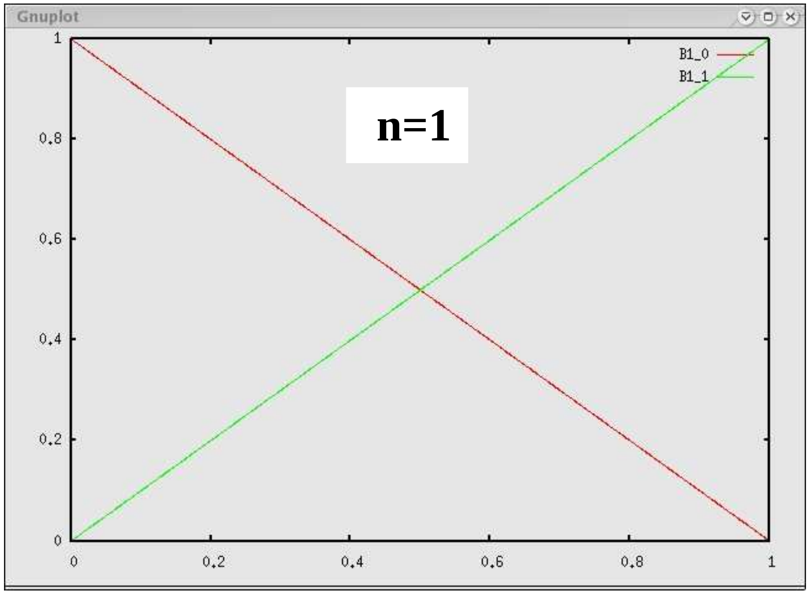
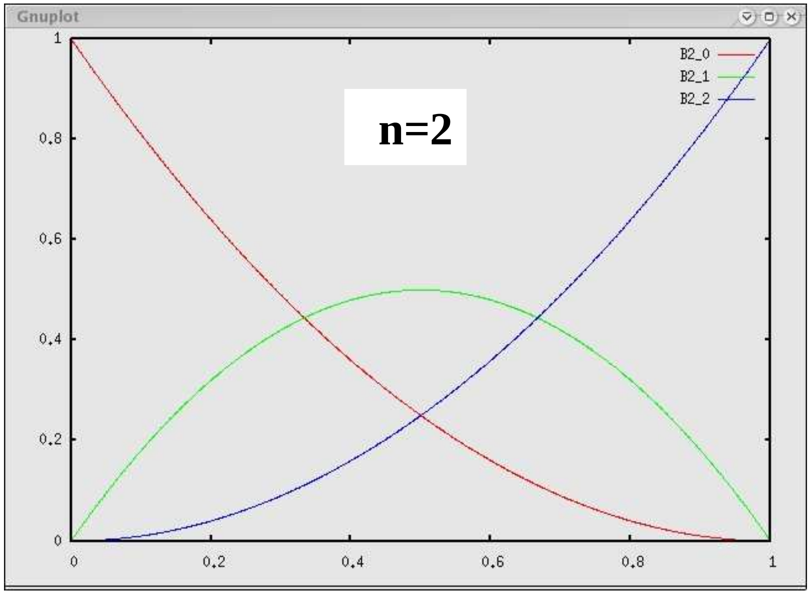
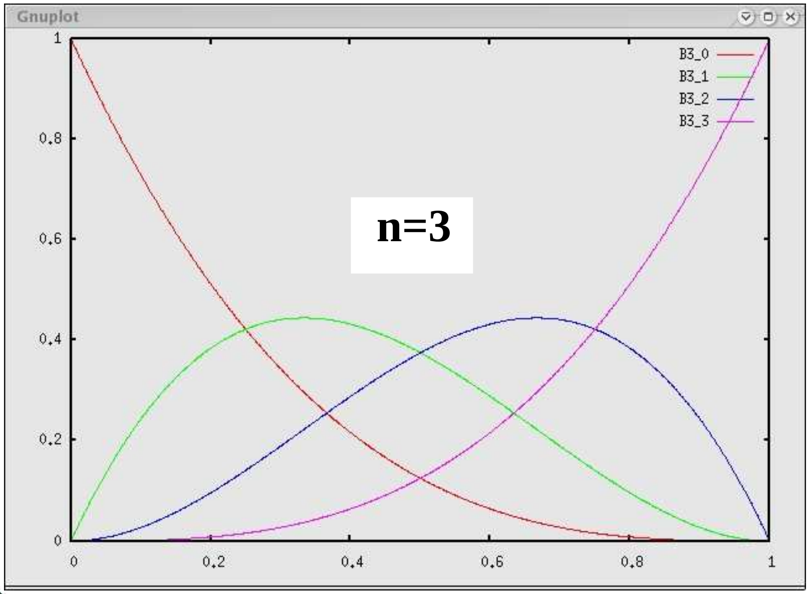
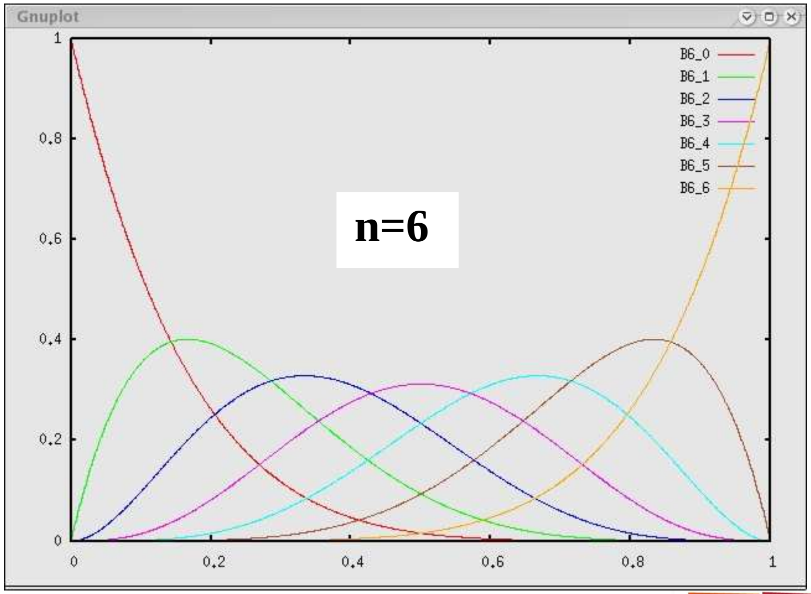
Plot of some Bernstein polynomials









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Properties

- Properties :
  - At a fixed degree, they are linearly independent,
  - They are symetric : $B_i^n(u) = B_{n-i}^n(1-u)$
  - They build a partition of unity : $\sum_{i=0}^{n} B_i^n(u) = 1 \qquad \forall u \in \mathbb{R}$
  - They are positive foa all u in [0,1] : $B_i^n(u) > 0 \qquad \forall u \in [0,1]$
  - They satisfy the recursive formula :

$$B_i^{n+1}(u) = u\, B_{i-1}^n(u) + (1-u)\, B_i^n(u)$$

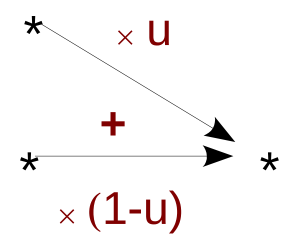
with : $B_{-1}^n = B_{n+1}^n = 0$ and $B_0^0 = 1$

Can be easily shown using : $\binom{n+1}{i} = \binom{n}{i-1} + \binom{n}{i}$

Compute Bernstein polynomials

- Triangular scheme :

$$
1 = \begin{array}{ccccc}
B_0^0 & B_0^1 & B_0^2 & \dots & B_0^n \\
 & B_1^1 & B_1^2 & \dots & B_1^n \\
 & & B_2^2 & \dots & B_2^n \\
 & & & \ddots & \vdots \\
 & & & & B_n^n
\end{array}
$$



- Exercise :
  - Compute Bernstein polynomials of degree 3.

Bézier curves

- Bézier curves :

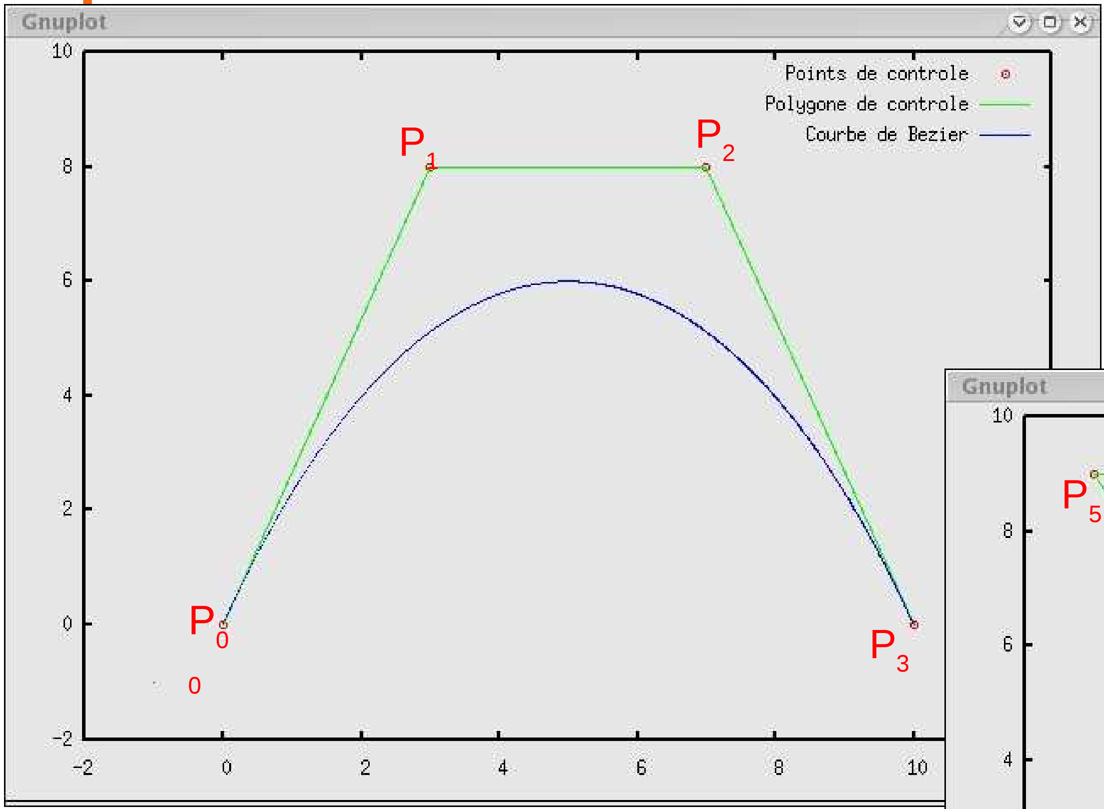
$$p(u)=\sum_{i=0}^{n} B_i^n(u)\,P_i \quad , \qquad u\in[0,1]$$

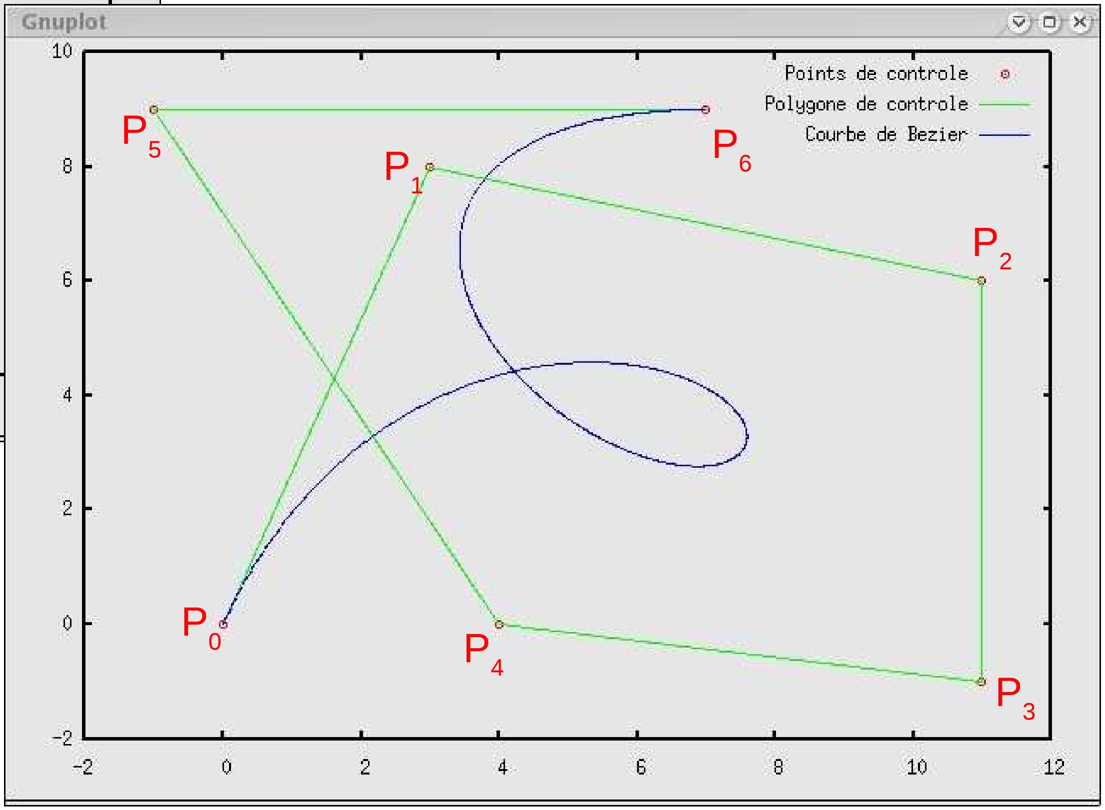
  - Points P_i (i=0..n) are the **n+1 control points** of the curve,
  - The curve is of **order n+1** and its **degree** is **n**,
  - The B_i^n areBernstein polynomials of degree n. They define the **basis functions** of the curve.

- The number of control points is directly linked to the curve degree : degree n ↔ n+1 control points.

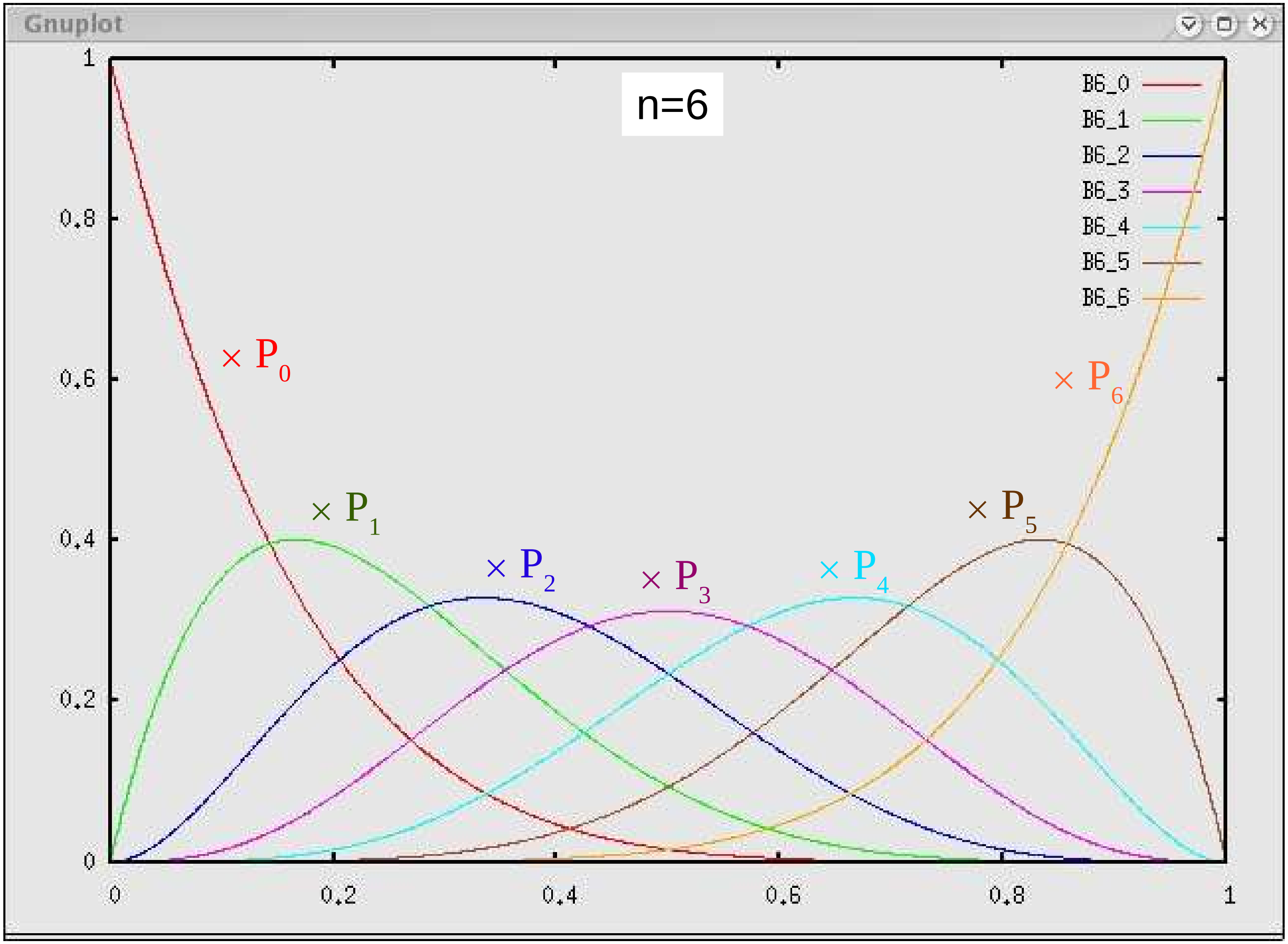
- Exercise :
  - A Bézier curve is controlled by the four points P_0 (0,0), P_1(5,5), P_2(10,5), P_3(15,0).
  - Compute p(0), p(1/4), p(1/2), p(3/4), p(1) with repsect to the P_i, then, compute the coordinates end plot the curve.

Examples of Bézier curves





Influence of the basis functions



Influence of the control points

- Exercise :
  - A Bézier curve is controlled by P_0 (0,0), P_1(5,5), P_2(10,5), P_3(15,0).

  Deduce from the value of the Bernstein polynomials the control points which the most influence the points on the curve for u=0, u=1/4, u=1/2, u=3/4, u=1.

Properties

- Symmetry :

$$p(u)=\sum_{i=0}^{n} B_i^n(u)\,P_i=\sum_{i=0}^{n} B_i^n(1-u)\,P_{n-i}$$

  - Thus, the curve remains the same whatever the ordering of the control points (0 to n or n to 0).

- Let t ∈ [a,b], t = a (1-u) + b.u, a≠b,

then :

$$p\left(t\left(u\right)\right)=p(t)=\sum_{i=0}^{n} B_i^n(u)\,P_i$$

- The Bézier curve interpolates its first and last control points (u ∈ [0,1]) :

$$p(0)=P_0 \qquad p(1)=P_n$$

- It is tangent to the first control and the last segment of its control polygon.

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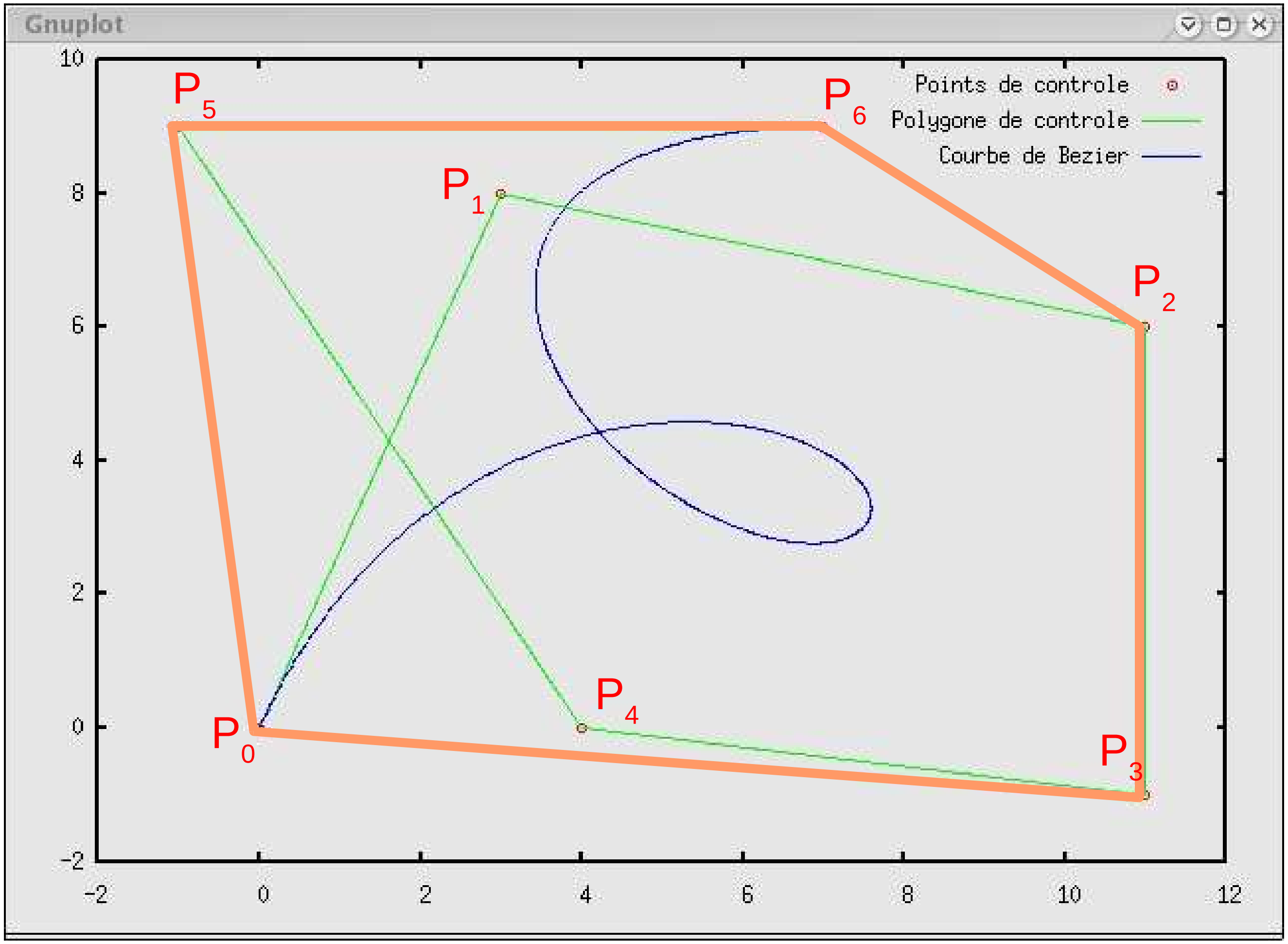
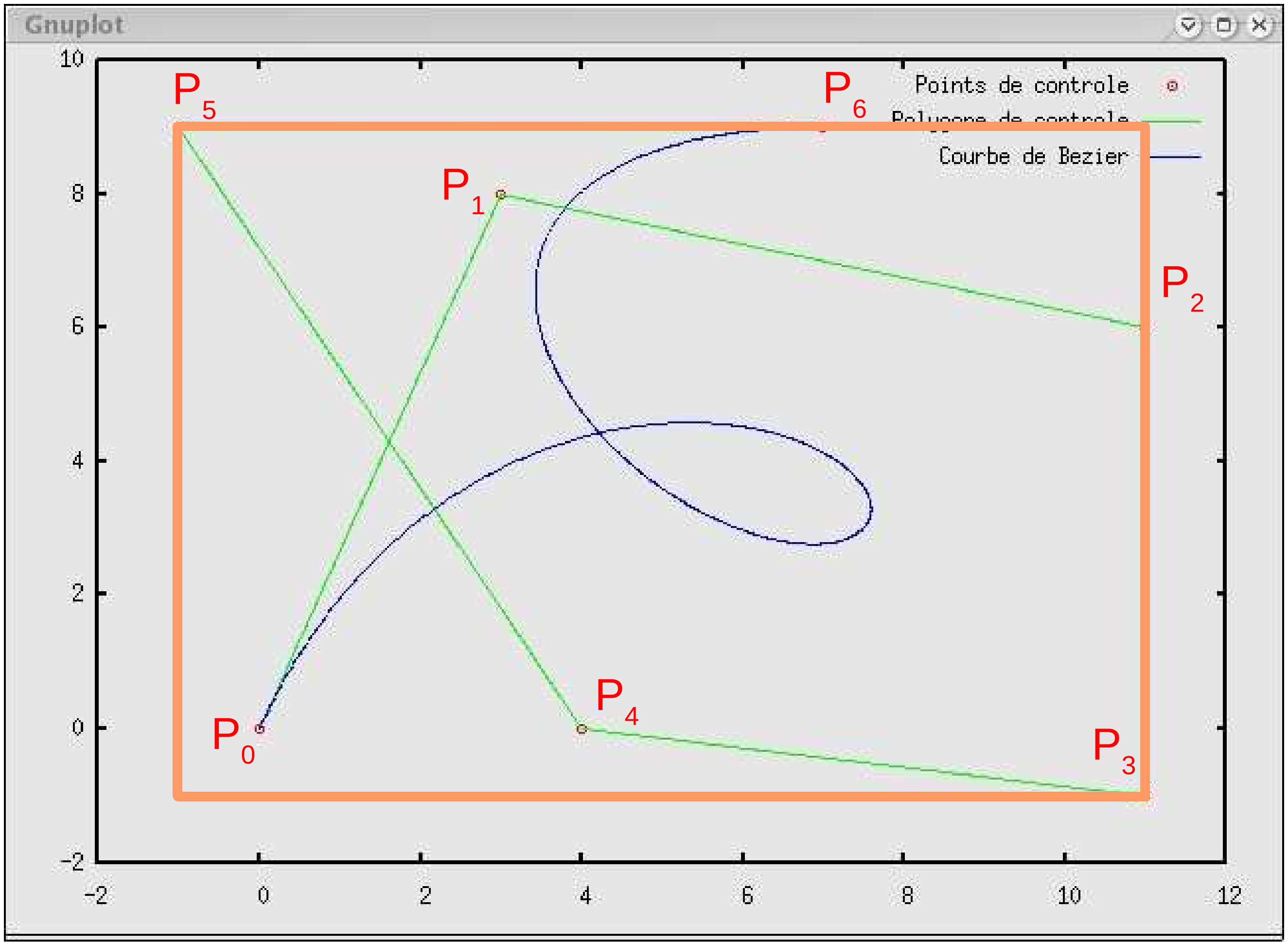
Convex hull

– The curve is included in the convex hull of its control polygon (because Bernstein polynomials are positive definite).



Bounding box

- The bounding box is obtained with the *min* and *max* of the coordinates of the control points. It is aligned with the frame axis.



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Derivative of a Bézier curve

- Derivative :
  - Show that : $$\frac{d}{du} B_i^n(u) = n\left(B_{i-1}^{n-1}(u) - B_i^{n-1}(u)\right)$$

- Deduce that :

$$\boxed{\frac{dp}{du}(u) = p^u(u) = n \sum_{i=0}^{n-1} B_i^{n-1}(u)\, \Delta P_i \quad , \qquad \Delta P_i = P_{i+1} - P_i}$$

- Note that :

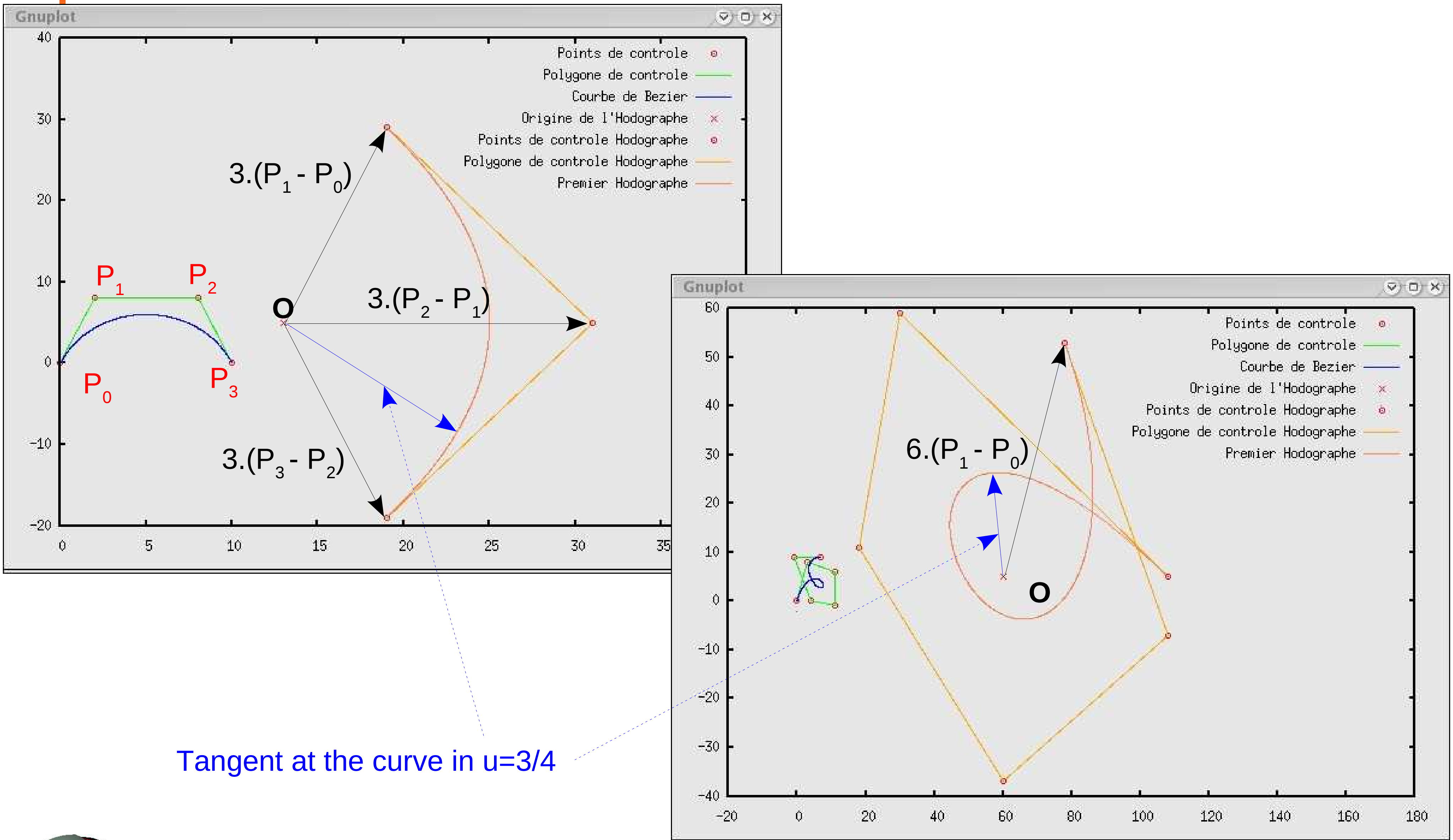
$$\frac{dp}{dt}(t) = p^t(t) = \frac{n}{b-a} \sum_{i=0}^{n-1} B_i^{n-1}(u)\, \Delta P_i \quad , \qquad t \in [a,b] \quad , \quad u \in [0,1]$$

- Exercise :
  - compute $p^u(0)$, $p^u(1/2)$, $p^u(1)$ with respect to points P_i for a Bézier curve of degree 3.

Hodograph

- Tangent vectors are computed as a « Bézier composition » of degree n-1 controled by vectors $V_i = n.\Delta P_i = n.(P_{i+1}-P_i)$, i=0..n-1 ($u \in [0,1]$).
- O is a point in space. The first hodograph of p(u) is the curve $O + p^u(u)$. The control polygon of this curve is defined by the points $O + V_i$

Examples of hodographs



Exercises

- Exercise :
  - Plot the first hodograph of Bézier curve controlled by P_0 (0,0), P_1(5,5), P_2(10,5), P_3(15,0)

- Exercise :
  - Give the matrix form of a Bézier curve of degree 3 :

  $p(u) = U . M . P$, where U is the matrix of the power of u, M is a squared matrix and P is the matrix of control points.

De Casteljau algorithm

- This algorithm relies on the following recursive formula ::

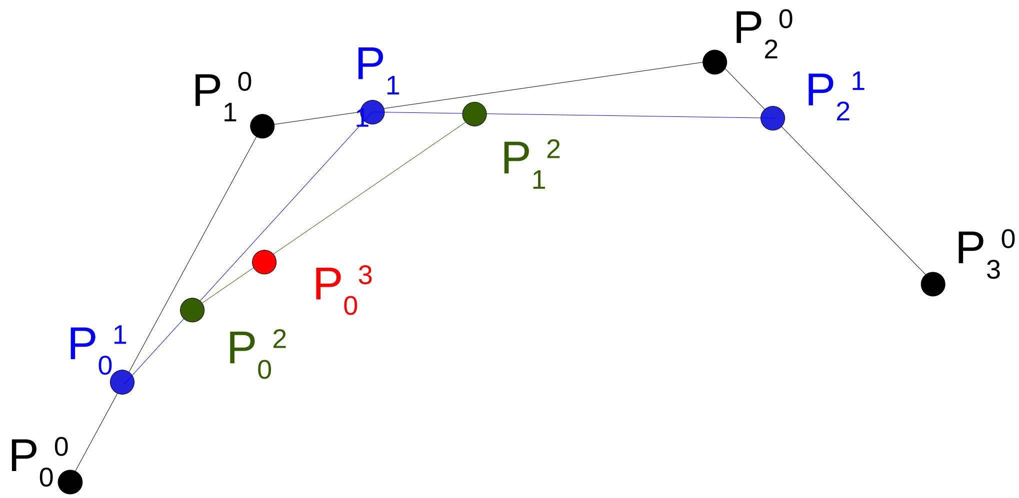
$$p(u)=\sum_{i=0}^{n} B_i^n(u)\,P_i^0=\sum_{i=0}^{n-1} B_i^{(n-1)}(u)\,P_i^1=\ldots=\sum_{i=0}^{0} B_i^0(u)\,P_i^n=P_0^n$$

where

$$P_i^{k+1}=(1-u)\,P_i^k+u\,P_{i+1}^k$$

Example with n=3 and u=1/4 :

P_0^3 is the point p(1/4)



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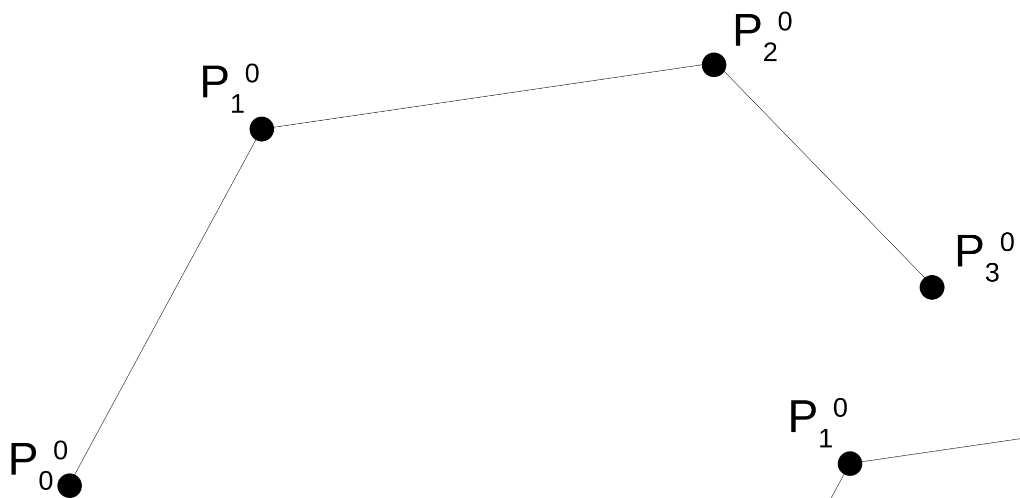
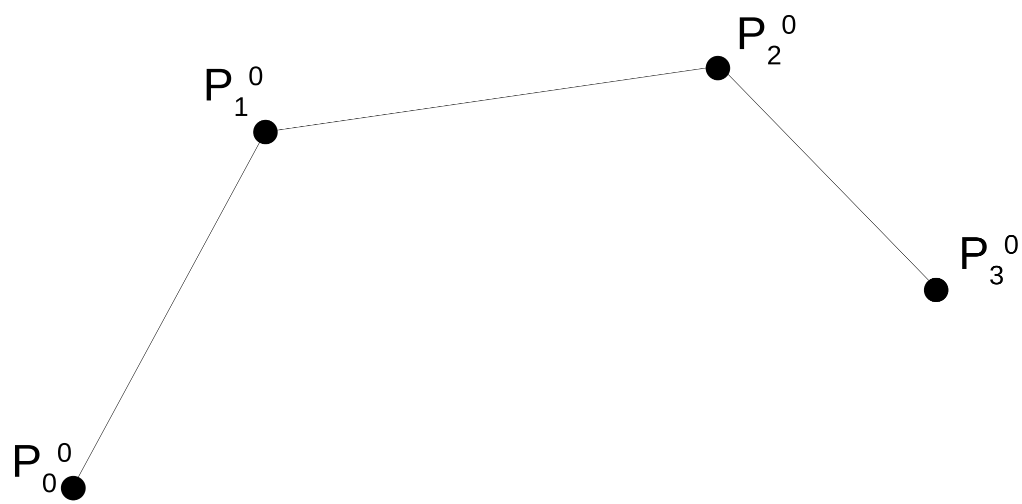
De Casteljau algorithm

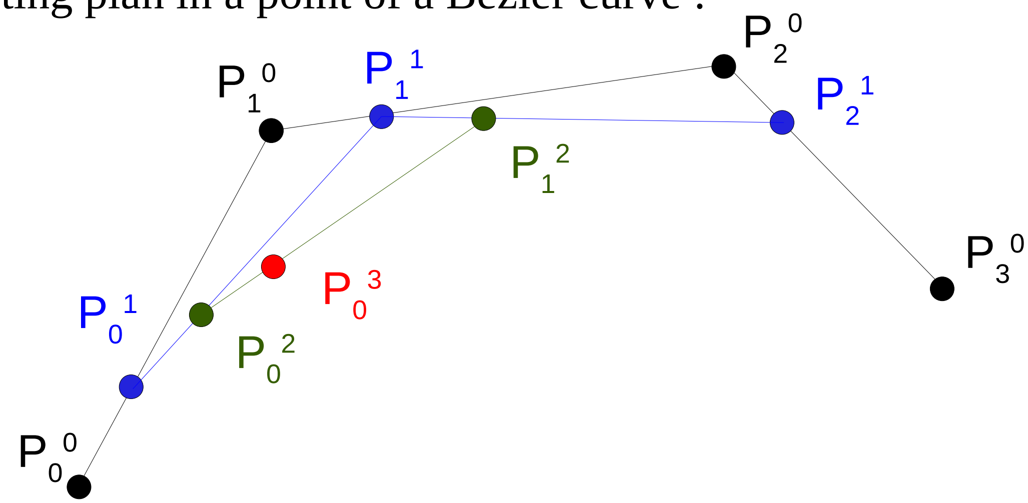
- Exercise :
  - Draw the computation of points p(1/2) and p(3/4) using the De Casteljau algorithm





De Casteljau algorithm

- Tangent and osculating plan in a point of a Bézier curve :



- The tangent in $P_0^3 = p(1/4)$ lies on the segment $[P_0^2, P_1^2]$
  - In général : $[P_0^{n-1}, P_1^{n-1}]$
- The osculating plan in P_0^3 is the plan (P_0^1, P_1^1, P_2^1)
  - In général : $(P_0^{n-2}, P_1^{n-2}, P_2^{n-2})$

Split in two Bézier curves

- The De Casteljau algorithm allows to split a control polygon composed of n+1 points in two control polygons of n+1 points each ::

$$p(u)=\sum_{i=0}^{n} B_i^n(u)\,P_i^0$$

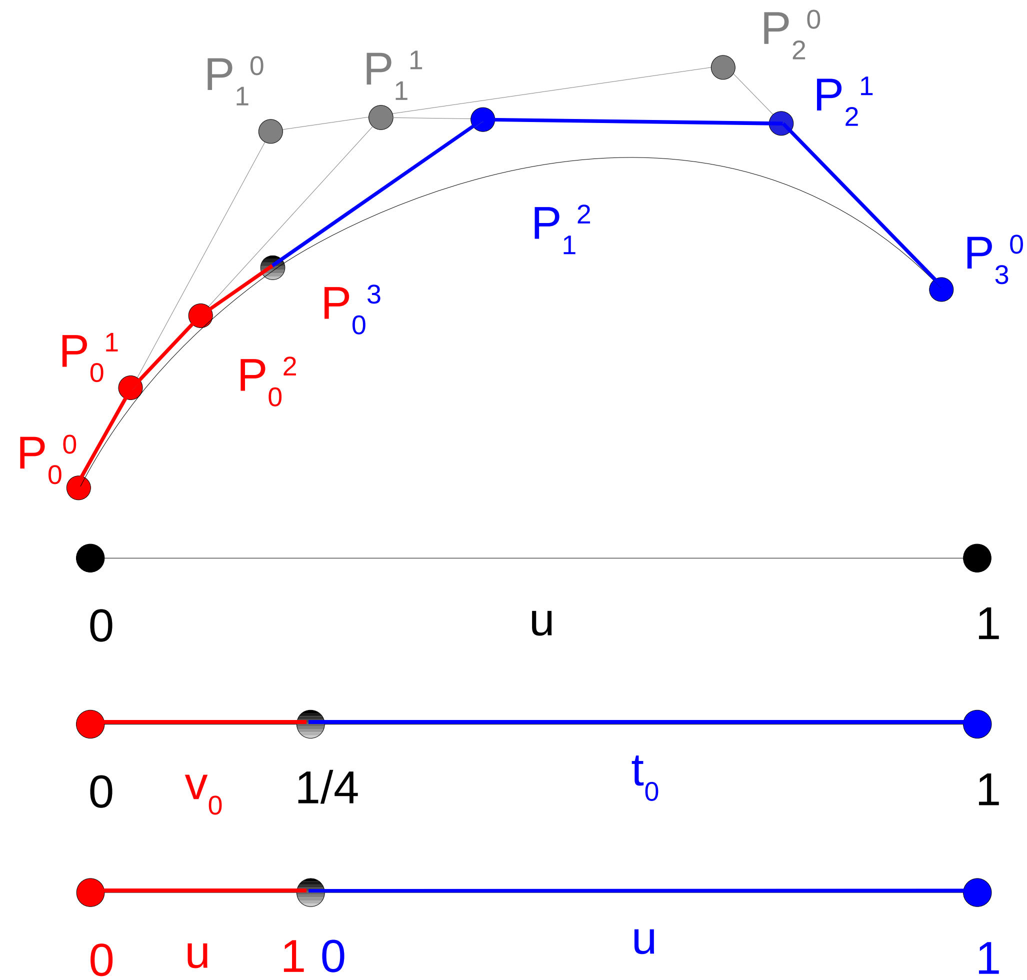
$$p_0(v_0)=\sum_{i=0}^{n} B_i^n(v_0)\,P_i \quad , \quad v_0\in\left[0,\frac{1}{4}\right]$$

$$p_0(v_1)=\sum_{i=0}^{n} B_i^n(v_0)\,P_i \quad , \quad v_1\in[0,1],\, v_0=\frac{1}{4}v_1$$

$$\boxed{p_0(u)=\sum_{i=0}^{n} B_i^n(u)\,P_0^i \quad , \quad u\in[0,1]}$$

$$p_1(t_0)=\sum_{i=0}^{n} B_i^n(t_0)\,P_i^0 \quad , \quad t_0\in\left[\frac{4}{3},1\right]$$

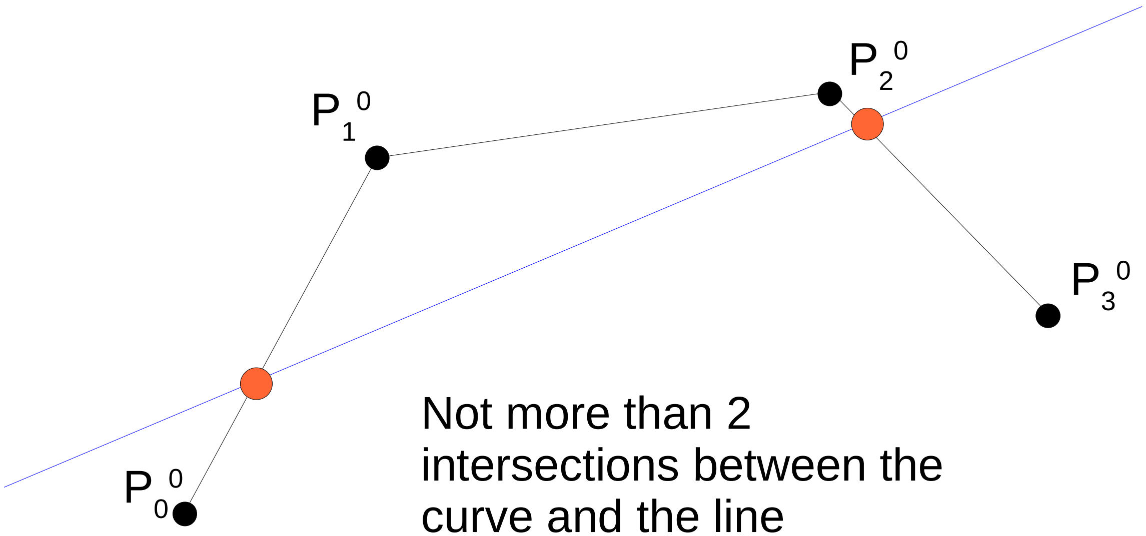
$$\boxed{p_1(u)=\sum_{i=0}^{n} B_i^n(u)\,P_i^{n-i} \quad , \quad u\in[0,1]}$$



VORTEX

Variational diminishing property

- A Bézier curve cannot have more intersection with a line than the maximum number of intersection between this line and the control polygon.

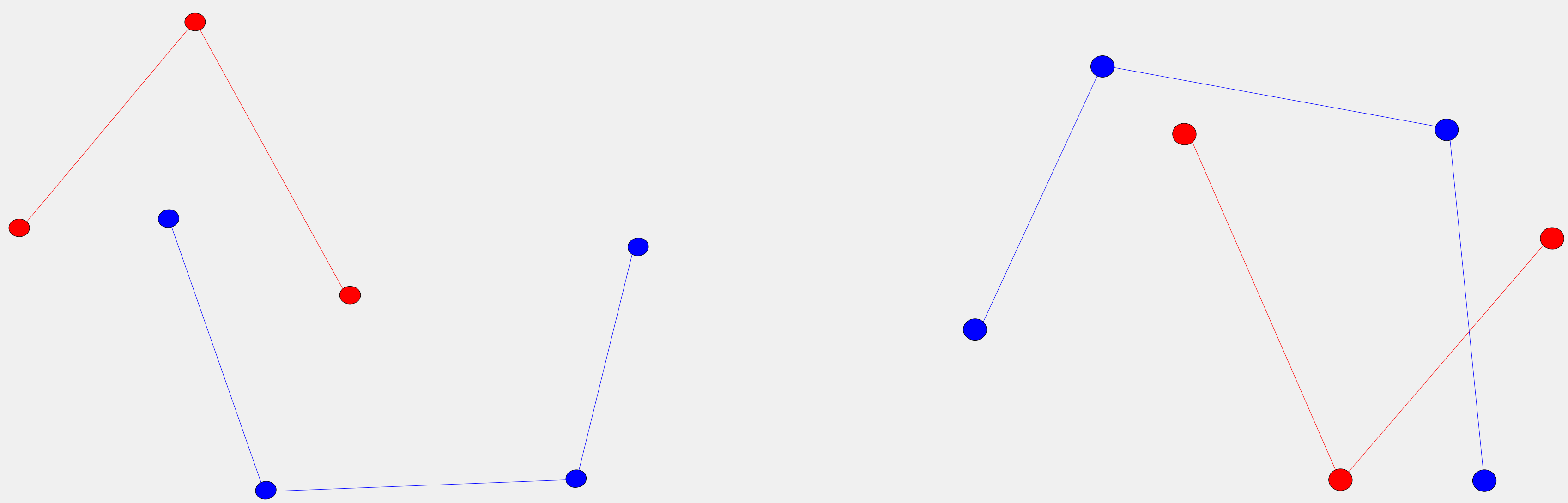


Not more than 2 intersections between the curve and the line

Intersections of two curves

- Intersections can be computed using the control polygons ::
  - If the bounding boxes of the control polygons intersects, each control polygon is subdivided with u=1/2 and the De Casteljau algorithm. Test intersections on the new polygons and subdivisde if there is an intersection. Repeat until a chosen precision is reached.

- Exercise :
  - Apply this algorithm on the following examples :



Increase the degree

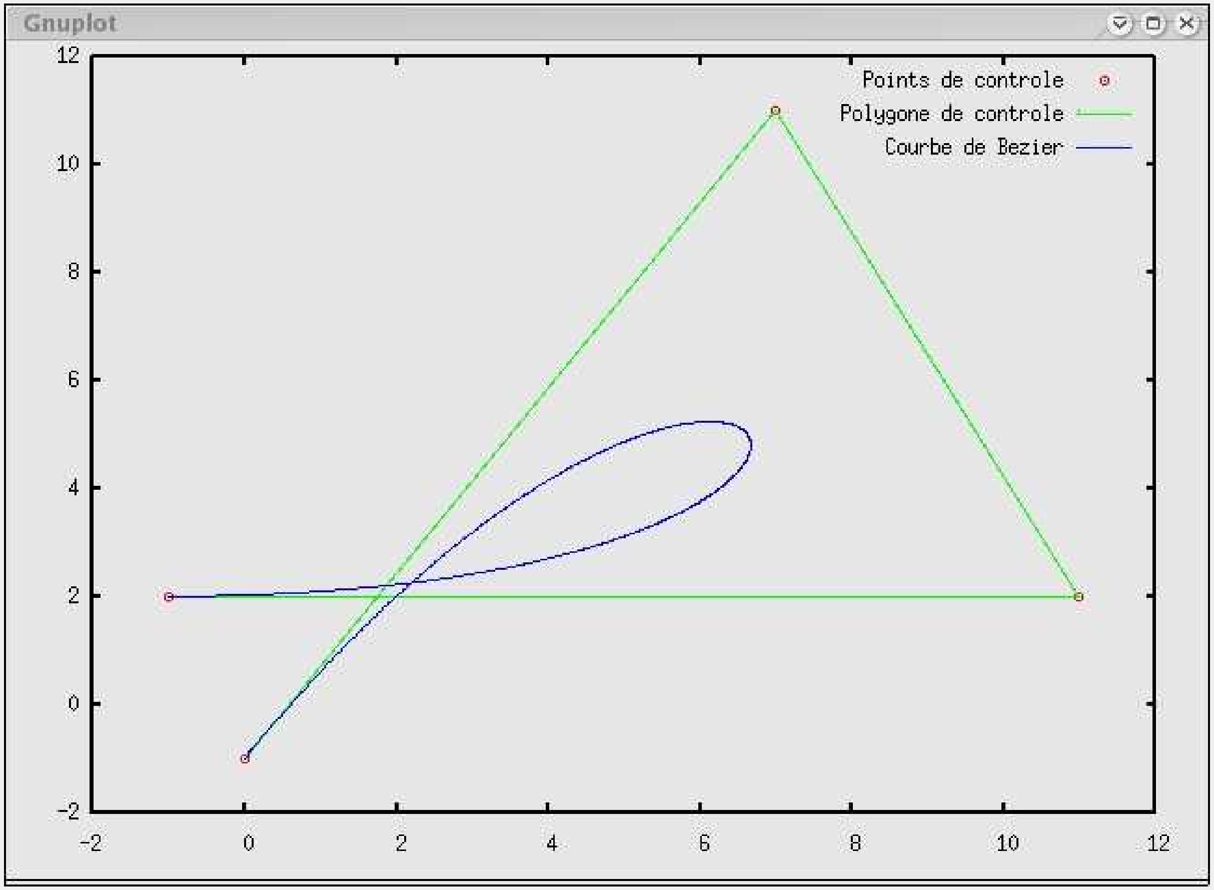
- A Bézier curve of degree n can always be represented with a Bézier curve of degree n+1.
- The new control points Q_i (i=0..n+1) are computed as follows ::

$$\boxed{Q_i = \frac{i}{n+1} P_{i-1} + \left(1 - \frac{i}{n+1}\right) P_i}$$

VORTEX

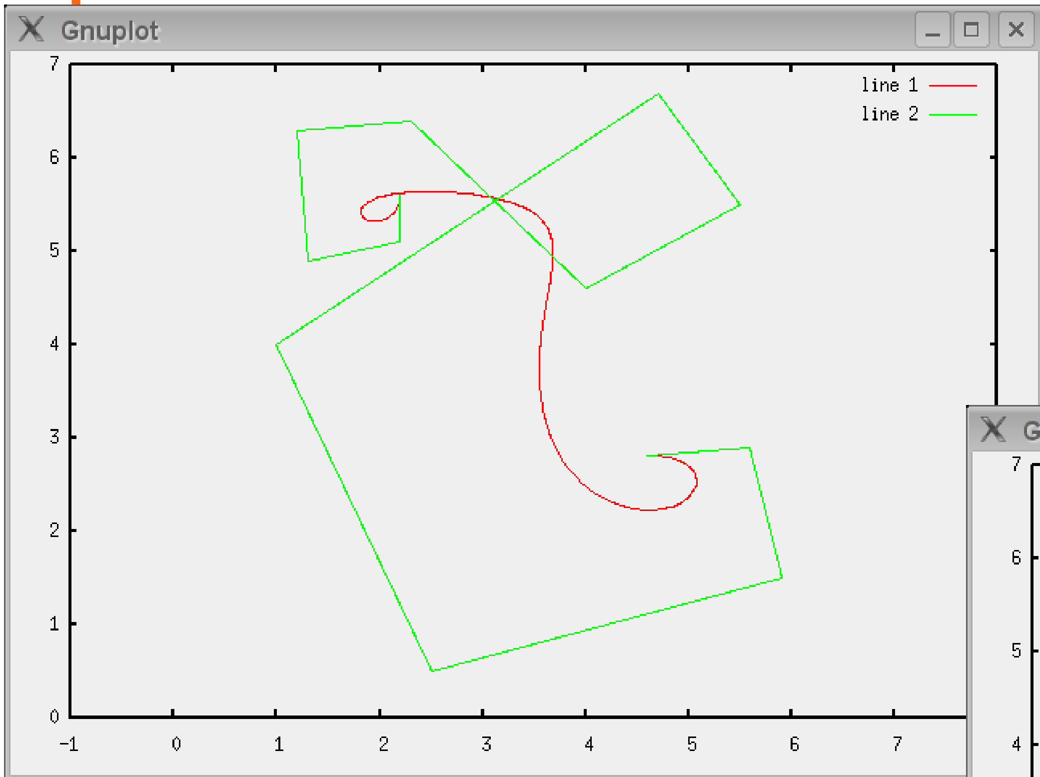
Increase the degree

- Exercise :
  - Apply the formula on the following example :
  - Check with the De Casteljau algorithm at u=(1/2) that p(1/2) is on the curve.

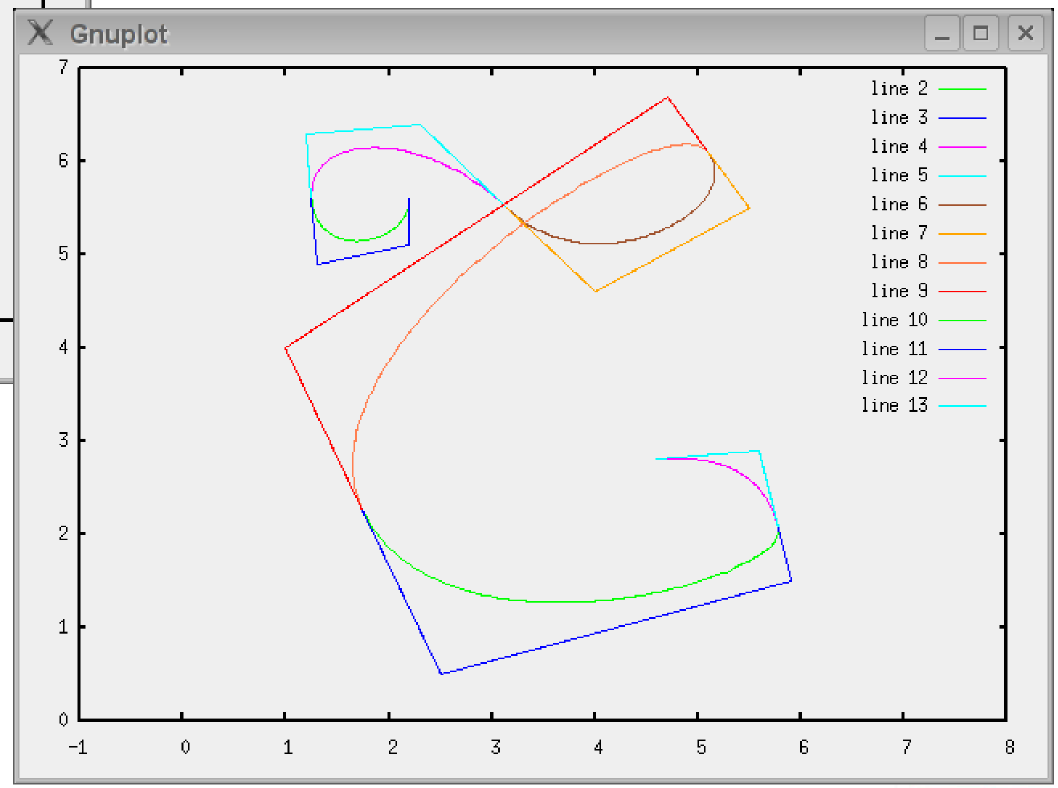


VORTEX

Modeling with Bézier curves



- A unique Bézier curve of degree 12



- 6 Bézier curves of degree 3 (degree 2 at the end) joined with a C^1 continuity

VORTEX

Join two curves

- Exercise :
  - p and q are two Bézier curves ::
    - $p(u)$ of degree n, $u \in [0,1]$, control points P_i
    - $q(v)$ of degree m, $v \in [0,1]$, control points Q_j
  - Give the join conditions in u=1 et v=0 in order to ensure the following continuities :
    - une continuité C^0
    - une continuité G^1
    - une continuité C^1
    - une continuité C^2

Strenghts and wiknesses

- Strenghts :
  - Intuitive control by control points
  - The curve is included in its convex hull
  - Easy to implement

- Désavantages :
  - Global support
  - The degree depends on the number of control points