# Some simple primitives

# Building a complex object

We might start:

- From the result of the 3D scan of an object. We have a starting model whose accuracy depends on the resolution of the 3D scanner.
- From planar 2D views of the object. We need at least 3 of these views to be able to rebuild its 3D shape.
- **From nothing**
- From drawings
- From a heightmap : the topography of some surface object is known

# Outline

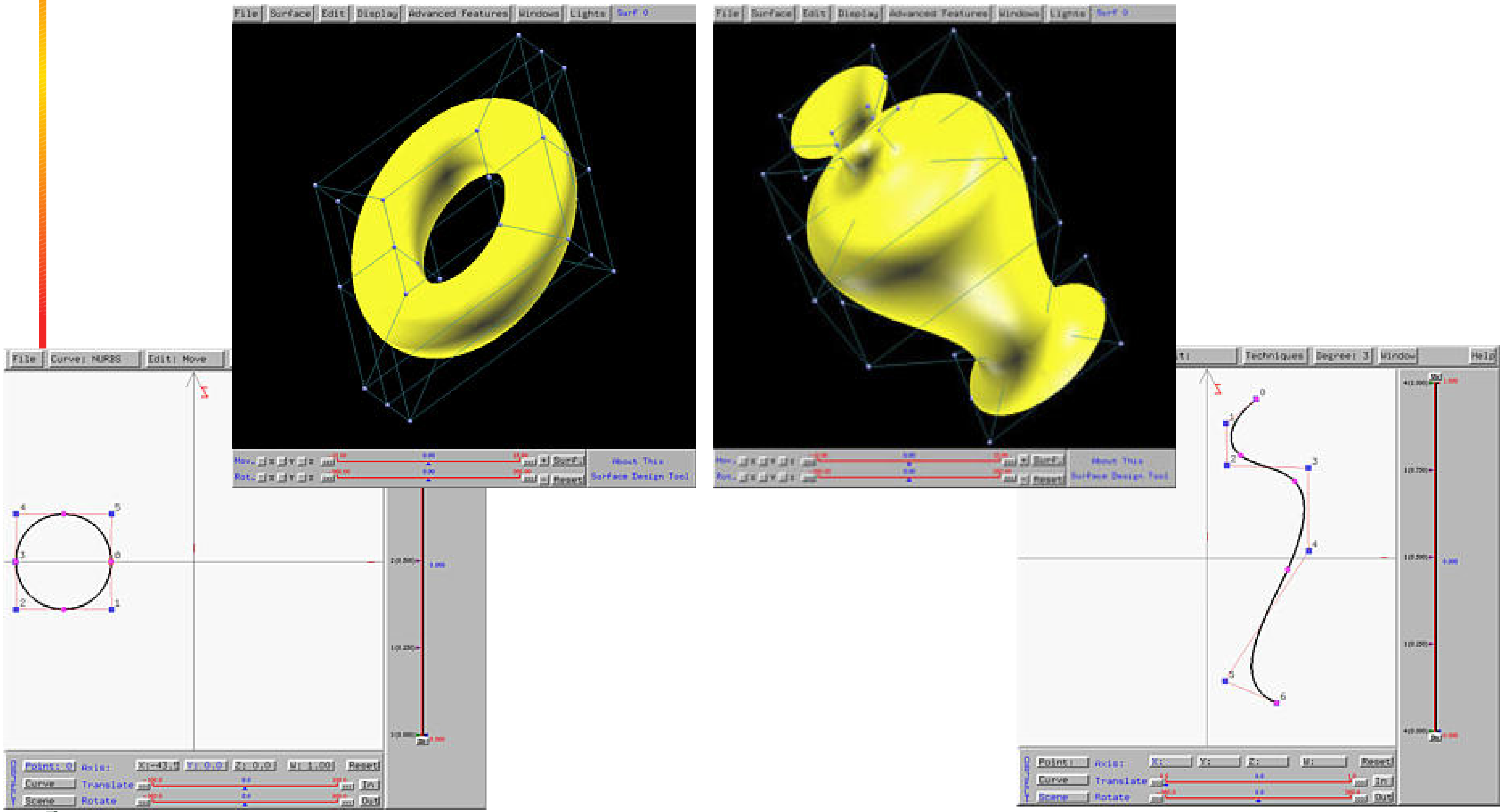
- Extrusion primitives
  - Revolution primitives
  - Sweeping primitives
- Main mathematic models
  - Implicit surfaces
  - Parametric surfaces
- Quadrics
  - The sphere
  - The cylinder
  - The cone
- Quartics
  - The torus
- Example: implicit and parametric equation of a surface obtained by extrusion

# Extrusion primitives

- 3D objects defined from a 2D profile (shape) and a trajectory.
- The profile can be defined from a mesh or an implicit or parametric equation.
- There are two kinds of extrusion primitives:
  - Revolution primitives
  - Sweeping primitives

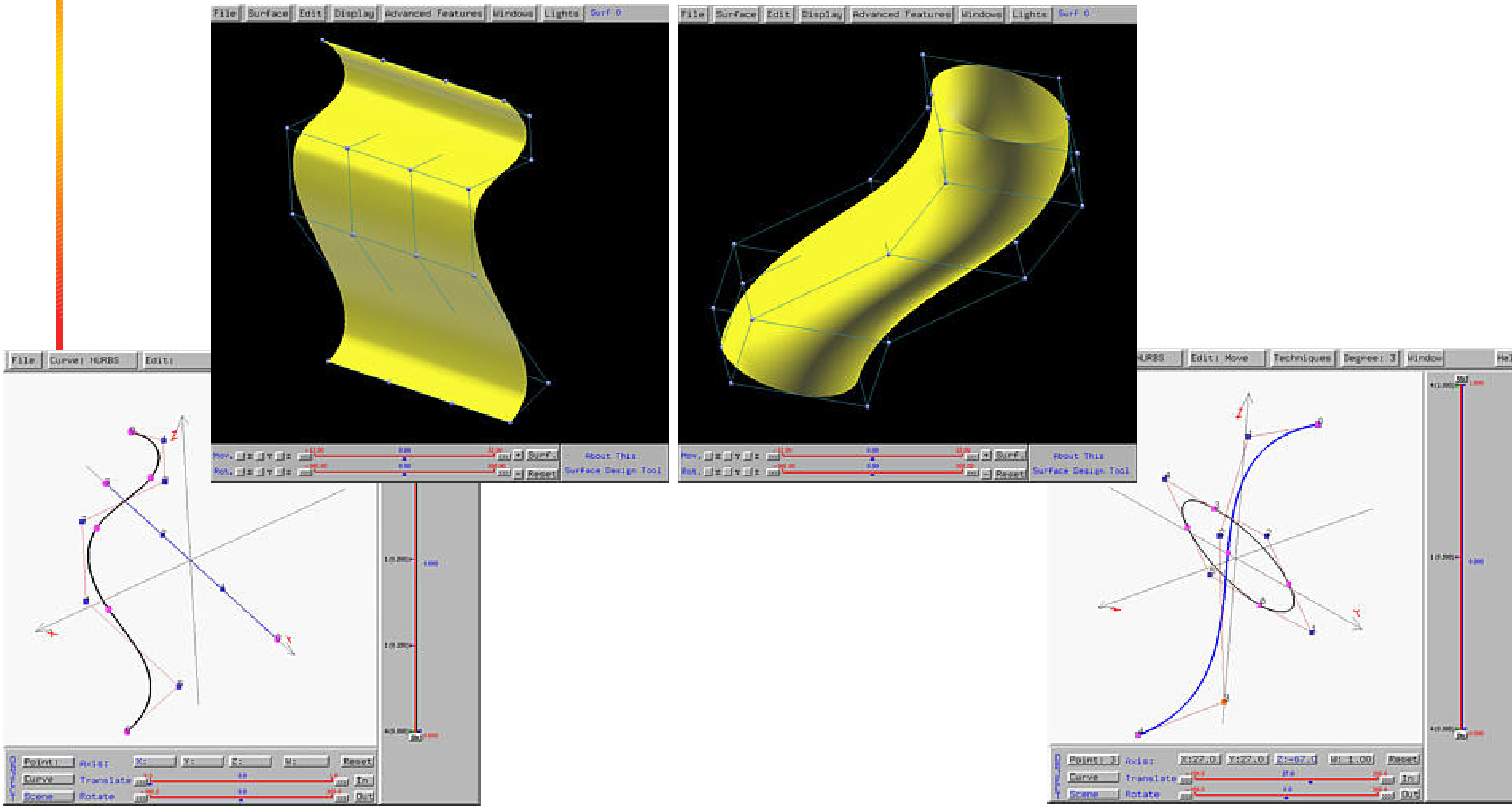
# Revolution primitives

- The 3D object is obtained by rotating the profile around some axis.

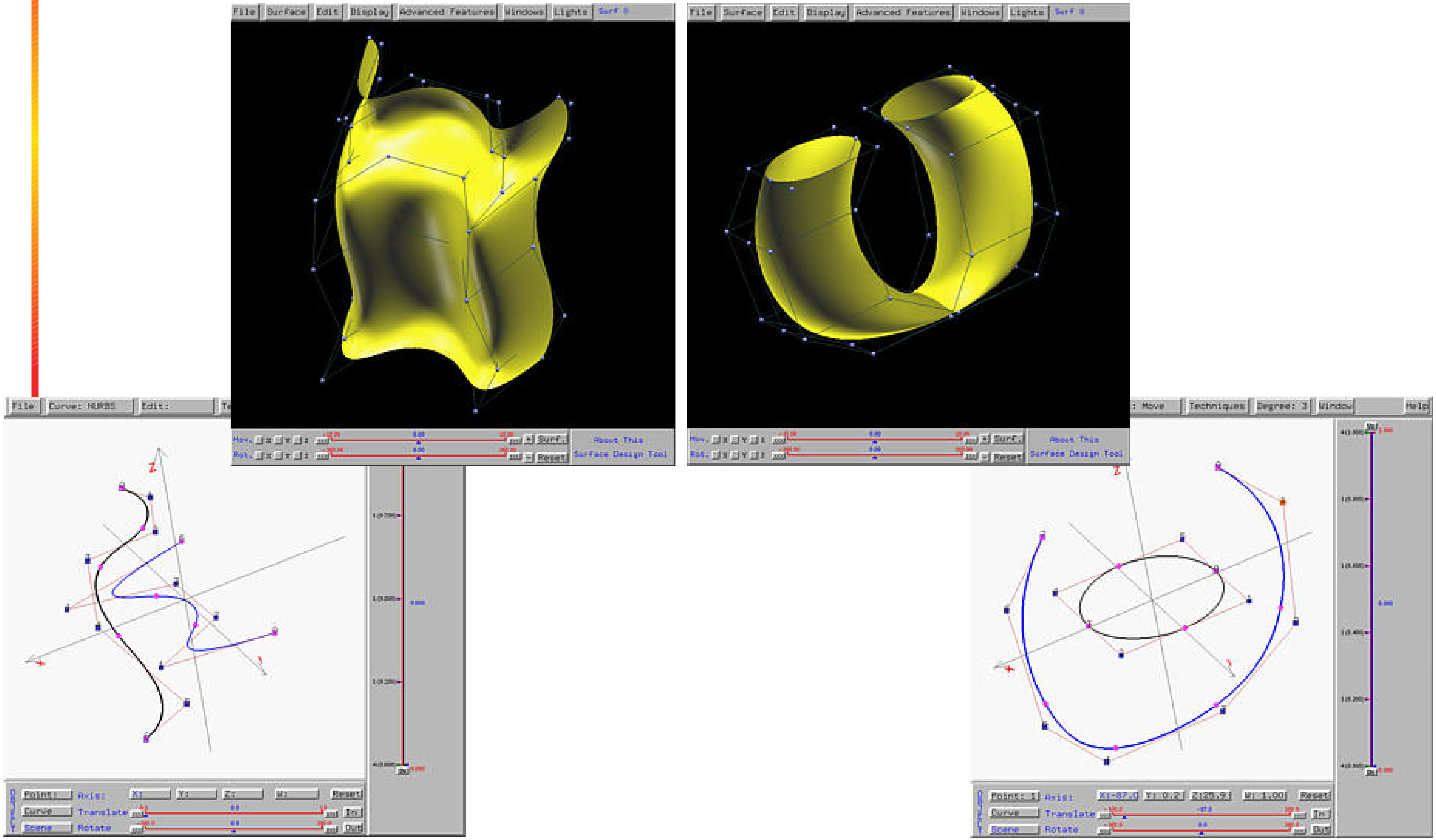


# Sweeping primitives

- The 3D object is obtained by « sweeping » the profile along some trajectory.



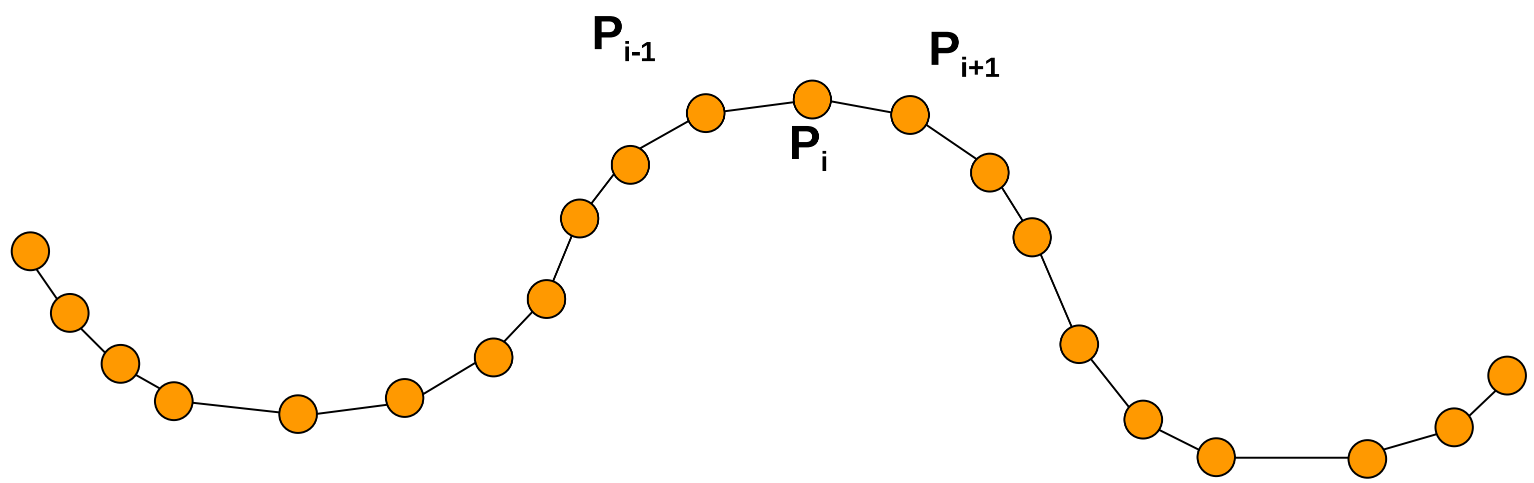
# Orientation issue



# Frenet frame (or TNB frame)

A local frame is computed from the explicit equation of the trajectory (if it exists) or from a discrete approximation of the trajectory.

Computing the tangent vector T using a discrete representation of the trajectory:




$T_i = P_{i+1} - P_i$ or $T_i = P_i - P_{i-1}$ or (better) $$\boxed{T_i = \frac{P_{i+1} - P_{i-1}}{2}}$$

# Plan normal

- Tangent vector : $\overline{T}_i = \dfrac{T_i}{\|T_i\|}$

- Tangent plane ($T_i$, $B_i$) : $\overline{B}_i = \dfrac{T_i \wedge A_i}{\|T_i \wedge A_i\|}$ with $A_i = \dfrac{T_{i+1} - T_{i-1}}{2}$

- Normal vector : $N_i = \overline{B}_i \wedge \overline{T}_i$

- Normal plane : ($N_i$, $B_i$)

- Caution: the orientation of the normal vector gets reverted at inflexion points

# Placing the profile on the Frenet frame

- Let the profile be defined on plane (xz). We set the center of the profile at point O (0,0,0). Let P be some point along the trajectory and $(T_P, N_P, B_P)$ the Frenet frame at point P
- What translation should be done in order to bring the center of the profile at point P.
- What rotation should we do so that the profile's plane is orthogonal to the tangent vector $T_P$ and so that the binormal vector $B_P$ matches the (Oz) axis in the initial profile plane (give the 3x3 rotation matrix).

# Mathematic models

- Two main classes :
  - Implicit surfaces
  - Parametric surfaces

# Implicit surfaces

- An implicit surface is defined from a potential function. A potential function is a continuous function $f:\mathbb{R}^3 \rightarrow \mathbb{R}$ that gives some potential value $f(P)$ for any point $P:(P_x, P_y, P_z)$ in $\mathbb{R}^3$ space. Given some threshold value T, the set of points $S\{P | f(P)=T\}$ defines a surface.
- The surface is defined implicitely : There is usually no way of computing explicitely the set of points S.
- On the other hand, the potential function completely defines the object's volume :
  - if f(P) > T, point P is outside the object
  - if f(P) < T, point P is inside the object
- A single surface usually have an infinite number of implicit forms. For instance given f and T, we can always find another potential function g that leads to the same surface but for the threshold value 0.

# Parametric surfaces

- A parametric surface is a function $f : D^2 \rightarrow \mathbb{R}^3$ where $D \subset \mathbb{R}$
- For each value $(u, v) \in D^2$, the point $P = f(u, v)$ belongs to the surface

$$\begin{cases} x_p = f_x(u, v) \\ y_p = f_y(u, v) \\ z_p = f_z(u, v) \end{cases}$$

- When u et v browse $D^2$, the associated points P browse the whole surface
- The surface is defined explicitely
- No notion of volume : we can't say whether a point is inside or outside the object

# Quadrics

- Quadrics are surfaces that can be computed from degree 2 polynomial equations:

$$f(x, y, z) = a_1 x^2 + a_2 xy + a_3 xz + a_4 x + a_5 y^2 + a_6 yz + a_7 y + a_8 z^2 + a_9 z + a_{10}$$

- A quadric equation can be written using a matrix:

$$f(x, y, z) = \begin{pmatrix} x & y & z & 1 \end{pmatrix} . \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$$f(x, y, z) = ax^2 + (b+e)xy + (c+i)xz + (d+m)x + fy^2 + (g+j)yz + (h+n)y + kz^2 + (l+o)z + p$$

# Symmetric matrix form

- Matrix forms that are optimized for matrix calculus:

$$f(x,y,z)=\begin{pmatrix} x & y & z & 1 \end{pmatrix} . \begin{pmatrix} a & \frac{b}{2} & \frac{c}{2} & \frac{d}{2} \\ \frac{b}{2} & e & \frac{f}{2} & \frac{g}{2} \\ \frac{c}{2} & \frac{f}{2} & h & \frac{i}{2} \\ \frac{d}{2} & \frac{g}{2} & \frac{i}{2} & j \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$$f(x,y,z)=ax^2+bxy+cxz+dx+ey^2+fyz+gy+hz^2+iz+j$$

# Triangular matrix form

$$f(x, y, z) = \begin{pmatrix} x & y & z & 1 \end{pmatrix} . \begin{pmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$$f(x, y, z) = ax^2 + bxy + cxz + dx + ey^2 + fyz + gy + hz^2 + iz + j$$

# The sphere

Sphere of radius r and center $(x_0,y_0,z_0)$ :

- Implicit form (with threshold value 0) :

$$f(x,y,z)=(x-x_0)^2+(y-y_0)^2+(z-z_0)^2-r^2$$

- Symmetric matrix form (implicit form) :

$$f(x,y,z)=\begin{pmatrix}x & y & z & 1\end{pmatrix}.\begin{pmatrix}1 & 0 & 0 & -x_0\\ 0 & 1 & 0 & -y_0\\ 0 & 0 & 1 & -z_0\\ -x_0 & -y_0 & -z_0 & (x_0^2+y_0^2+z_0^2-r^2)\end{pmatrix}.\begin{pmatrix}x\\ y\\ z\\ 1\end{pmatrix}$$

# The sphere

- Parametric form :

$$f(\varphi,\theta)=\begin{cases} x_0+r\cos\varphi\cos\theta \\ y_0+r\sin\varphi\cos\theta \\ z_0+r\sin\theta \end{cases} \qquad 0\leq\varphi<2\pi\,,\ -\frac{\pi}{2}\leq\theta<\frac{\pi}{2}$$

# The cylinder

Vertical cylinder (z axis) with radius r and center $(x_0,y_0,0)$ (infinite cylinder)

- Implicit form :

$$f(x,y,z)=(x-x_0)^2+(y-y_0)^2-r^2$$

- Symmetric matrix form (implicit form) :

$$f(x,y,z)=\begin{pmatrix}x & y & z & 1\end{pmatrix}.\begin{pmatrix}1 & 0 & 0 & -x_0\\ 0 & 1 & 0 & -y_0\\ 0 & 0 & 1 & 0\\ -x_0 & -y_0 & 0 & (x_0^2+y_0^2-r^2)\end{pmatrix}.\begin{pmatrix}x\\ y\\ z\\ 1\end{pmatrix}$$

# The cylinder

- Parametric form :

$$f(\varphi, h) = \begin{cases} x_0 + r\cos\varphi \\ y_0 + r\sin\varphi \\ h \end{cases} \qquad 0 \le \varphi < 2\pi,\ h \in \mathbb{R}$$

# The cone

Vertical cone (z axis) with apex $(x_0,y_0,z_0)$ and aperture $\alpha$ : (infinite cone)

- Implicit form :

$$f(x,y,z)=(x-x_{.})^2+(y-y_0)^2-\tan^2\alpha(z-z_0)^2$$

- Symmetric matrix form (implicit form) :

$$f(x,y,z)=\begin{pmatrix}x & y & z & 1\end{pmatrix}.\begin{pmatrix}1 & 0 & 0 & -x_0\\ 0 & 1 & 0 & -y_0\\ 0 & 0 & -\tan^2\alpha & z_0\tan^2\alpha\\ -x_0 & -y_0 & z_0\tan^2\alpha & (x_0^2+y_0^{٢}-z_0^2\tan^2\alpha)\end{pmatrix}.\begin{pmatrix}x\\ y\\ z\\ 1\end{pmatrix}$$

# The cone

- Parametric form :

$$f(\varphi, h) = \begin{cases} x_0 + h \tan\alpha \cos\varphi \\ y_0 + h \tan\alpha \sin\varphi \\ z_0 + h \end{cases} \qquad 0 \leq \varphi < 2\pi,\ h \in \mathbb{R}$$

# Quartics

- Quadrics are surfaces that can be computed from degree 4 polynomial equations:

$$f(x, y, z) = a_1 x^4 + a_2 x^3 y + a_3 x^3 z + a_4 x^2 y^2 + \ldots + a_{33} z + a_{34}$$

- They do not have a matrix form

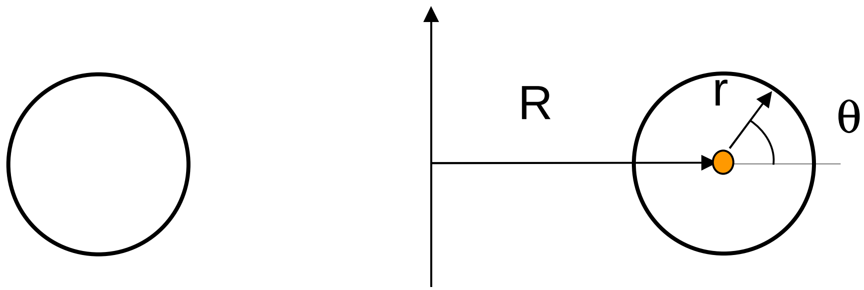
# The torus

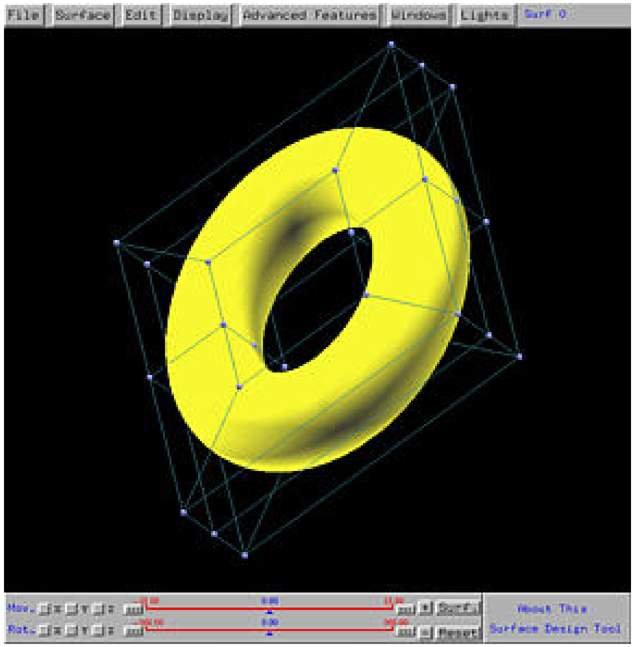
- The implicit polynomial form is not very common:

$$f(x,y,z)=(x^2+y^2+z^2+R^2-r^2)^2-4R^2(x^2+y^2)$$

- Parametric form :

$$f(\varphi,\theta)=\begin{cases}(R+r\cos\theta).\cos\varphi\\(R+r\cos\theta).\sin\varphi\\r\sin\theta\end{cases}\qquad 0\leq\varphi,\theta<2\pi$$



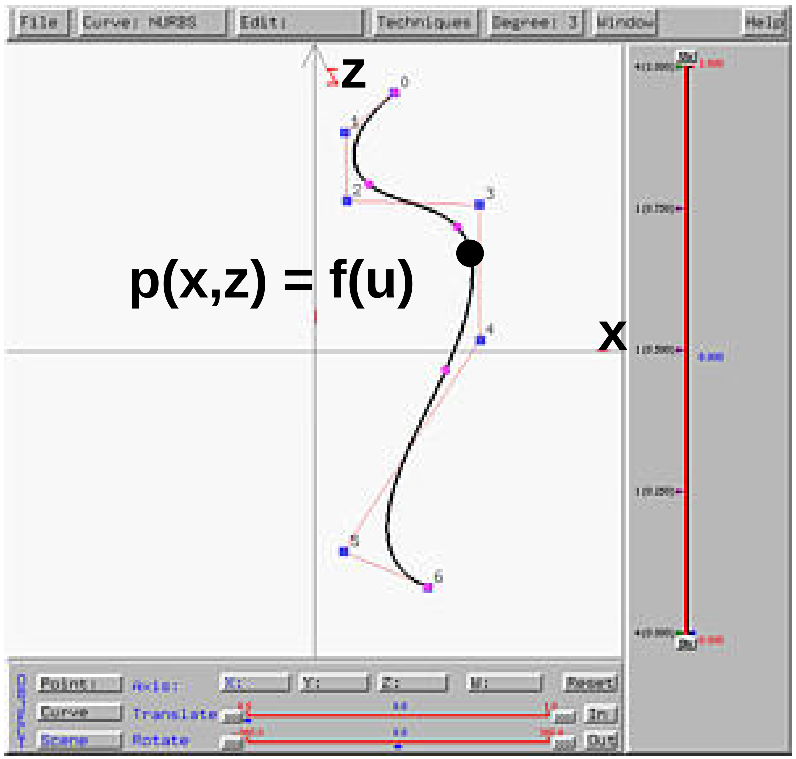



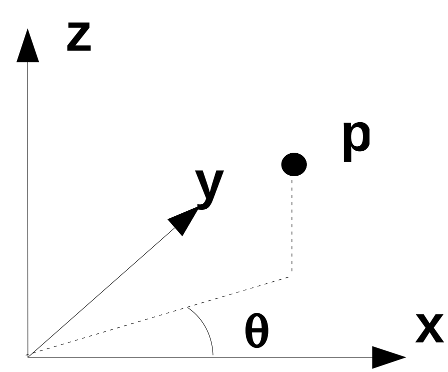
# Revolution surface

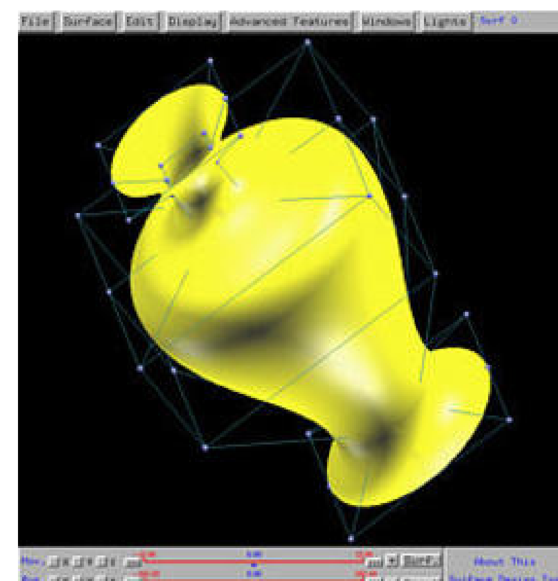
- Profile defined by a parametric equation
  - Let f(u), u ∈ [a,b], be the parametric equation of the profile in plane (x,z), and θ the angle between the (Ox) axis and point p in plane (x,y) :

$$p(x,z) = f(u) = \begin{cases} x = f_x(u) \\ z = f_z(u) \end{cases} \quad \Rightarrow \quad p(x,y,z) = g(u,\theta) = \begin{cases} x = f_x(u)\cos\theta \\ y = f_x(u)\sin\theta \\ z = f_z(u) \end{cases}$$

$$\theta \in [0,2\pi]$$



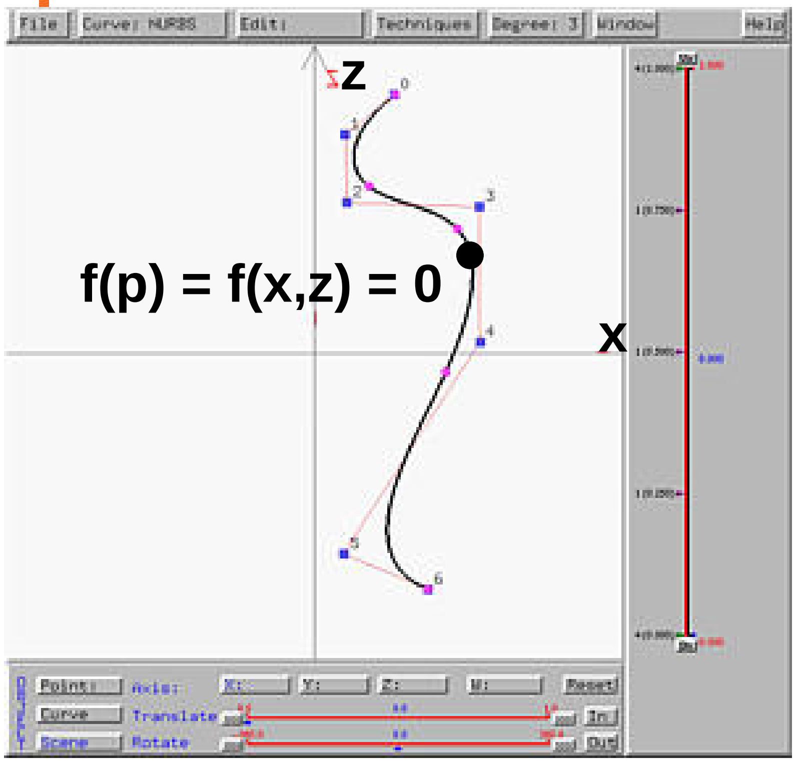
# Surface de révolution

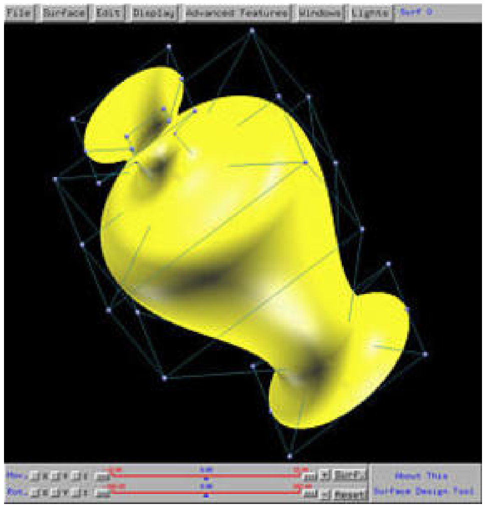
- Profile defined by an implicit equation
  - Let f : $\mathbb{R}^2 \rightarrow \mathbb{R}$ be a potential function such that the profile is defined by :

$$profile = \{ p(x,z) \in \mathbb{R}^2 / f(p) = 0 \}$$

  - Let **d** be the euclidean distance between point p(x,y,z) and the (Oz) axis.
  - The revolution surface is defined by:

$$S = \{ p(x,y,z) \in \mathbb{R}^3 / g(p) = f(d,z) = 0 \}$$



# Tutorial: cubic Hermite spline

- Given two points P0, P1 and two velocity vectors V0 and V1, the cubic Hermite spline of those parameters is the 3rd degree polynomial parametric curve $p(t)=A.t^3+B.t^2+C.t+D$ $t\in[0;1]$ such that:
  - $p(0)=P0$
  - $p(1)=P1$
  - $\frac{d}{dt}p(0)=P$
  - $\frac{d}{dt}p(1)=P1$

- Express the 4 parameters A, B, C, D with P0, P1, V0, V1
- Put the equation under the form $p(t)=a(t).P0+b(t).P1+c(t).V0+d(t).V1$ with a, b, c and d being 3rd degree polynoms.
- Show that p(t) can be put under matrix form:

$$p(t)=(t^3 \quad t^2 \quad t \quad 1).\begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix}.\begin{pmatrix} P0 \\ P1 \\ V0 \\ V1 \end{pmatrix}$$